Chapter 1
Units and Measurement

Introduction

Measurement of any physical quantity involves comparison with a certain basic, internationally accepted reference standard called unit.
The result of a measurement of a physical quantity is expressed by a number accompanied by a unit.

Fundamental and Derived Quantities.

The quantities ,which can be measured directly or indirectly are called physical quantities. There are two types of physical quantities- Fundamental quantities(Base quantities) and Derived quantities.

- **The physical quantities, which are independent of each other and cannot be expressed in terms of other physical qualities are called fundamental quantities.**
  Eg: length, mass, time.
- **The physical quantities , which can be expressed in terms of fundamental qualities are called derived quantities.**
  Eg: volume, velocity, force

Fundamental and Derived Units

- The units for the fundamental or base quantities are called fundamental or base units. The units of all other physical quantities can be expressed as combinations of the base units.
- The units of the derived quantities are called derived units.

Systems of Units

A complete set of both the base and derived units, is known as the system of units. Three such systems, the CGS, the FPS (or British) system and the MKS system were in use extensively till recently.
The base units for length, mass and time in these systems were as follows :

- CGS system - centimetre, gram and second.
- FPS system - foot, pound and second.
- MKS system - metre, kilogram and second.

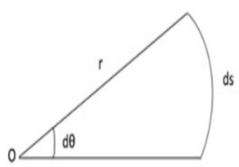
The International System of Units

- In 1971 the General Conference on Weights and Measures developed an internationally accepted system of units for measurement with standard scheme of symbols, units and abbreviations.
- **This is the Système Internationale d' Unites (French for International System of Units), abbreviated as SI system.**
- SI system is now for international usage in scientific, technical, industrial and commercial work.

In SI system there are seven base units and two supplementary units.

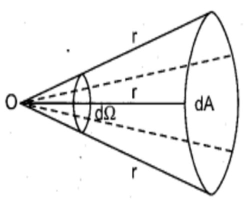
BASE QUANTITY	BASE UNIT	SYMBOL
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric Current	ampere	A
Thermodynamic Temperature	kelvin	K
Amount of Substance	mole	mol
Luminous Intensity	candela	cd

SUPPLEMENTARY QUANTITY	SUPPLEMENTARY UNITS	SYMBOL
Plane Angle	radian	rad
Solid Angle	steradian	sr



$$\text{Angle} = \frac{arc}{radius}$$

$$d\theta = \frac{ds}{r}$$



$$\text{Solid angle} = \frac{\text{Intercepted } Area}{Square\ of radius}$$

$$d\Omega = \frac{dA}{r^2}$$

Multiples and Sub multiples of Units

Submultiple	Prefix	Symbol	Multiple	Prefix	Symbol
10^{-1}	deci	d	10	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P

Dimensions, Dimensional Formula and Dimensional Equation

The volume occupied by an object = length x breadth x height
The dimensions of volume represented as [V]

$$[V] = [L] \times [L] \times [L] = [L]^3 = [L^3].$$

$$[V] = [L^3]$$

$$[V] = [M^0 L^3 T^0]$$

Dimensions

Dimensions of a physical quantity are the powers (or exponents) to which the base quantities are raised to represent that quantity.

Volume has 0 Dimensions in mass, 3 Dimensions in length and 0 Dimensions in Time

Dimensional Formula

The expression which shows how and which of the base quantities represent the dimensions of a physical quantity is called the dimensional formula of the given physical quantity.

The dimensional formula of volume [V] is expressed as $[M^0 L^3 T^0]$

Dimensional Equation

An equation obtained by equating a physical quantity with its dimensional formula is called the dimensional equation of the physical quantity.

$$[V] = [M^0 L^3 T^0]$$

Dimensional Formula of some derived quantities

1. Area = Length x Breadth

$$[A] = [L] \times [L]$$

$$[A] = [L^2]. \quad \text{Or} \quad [A] = [M^0 L^2 T^0]$$

Unit of area = m^2

2. Volume = Length x Breadth x Height

$$[V] = [L] \times [L] \times [L]$$

$$[V] = [L^3]$$

Unit of volume = m^3

3. Density = $\frac{Mass}{Volume}$

$$[\rho] = \frac{[M]}{[L^3]}$$

$$[\rho] = [ML^{-3}].$$

Unit of density = kg m^{-3}

4 .Frequency $= \frac{1}{Time\ period}$

$$[f] = \frac{1}{[T]}$$

$$[f] = [T^{-1}]$$

Unit of frequency $= s^{-1}$

5. Speed $= \frac{\text{Distance}}{\text{time}}$

$$[s] = \frac{[L]}{[T]}$$

$$[s] = [LT^{-1}]$$

Unit of speed $= m\ s^{-1}$

6. Velocity $= \frac{\text{Displacement}}{\text{time}}$

$$[v] = \frac{[L]}{[T]}$$

$$[v] = [LT^{-1}]$$

Unit of velocity $= m\ s^{-1}$

Speed and Velocity have same dimensional formula

7. Momentum = Mass x Velocity

$$[p] = [M] \times [LT^{-1}]$$

$$[p] = [MLT^{-1}]$$

Unit of momentum $=$ kg m s^{-1}

8. Angular Momentum = momentum x Distance

$$[L] = [MLT^{-1}] \times [L]$$

$$[L] = [M L^2 T^{-1}]$$

Unit of angular momentum $=$ kg $m^2\ s^{-1}$

9. Acceleration = $\frac{\text{Change in velocity}}{\text{time}}$

$$[a] = \frac{[LT^{-1}]}{[T]}$$

$$[a] = [LT^{-1}] \times [T^{-1}]$$

$$[a] = [LT^{-2}]$$

Unit of acceleration = $m\ s^{-2}$

10. Force = Mass x Acceleration

$$[F] = [M] \times [LT^{-2}]$$

$$[F] = [MLT^{-2}]$$

Unit of force = $kg\ m\ s^{-2}$ or newton(N)

$1 kgms^{-2} = 1N$

11.Impulse = Force x Time

$$[I] = [MLT^{-2}] \times [T]$$

$$[I] = [MLT^{-1}]$$

Unit of Impulse = $kg\ m\ s^{-1}$

12. Work = Force x Displacement

$$[W] = [MLT^{-2}] \times [L]$$

$$[W] = [M L^2 T^{-2}]$$

Unit of work = $kg\ m^2\ s^{-2}$ or joule (J)

$1kgm^2 s^{-2}$ =1J

13. Energy = Workdone

$$[E] = [M L^2 T^{-2}]$$

Unit of energy = $kg\ m^2\ s^{-2}$ or joule (J)

14. Torque = Force x perpendicular Distance

$$[\tau] = [MLT^{-2}] \times [L]$$

$$[\tau] = [M L^2 T^{-2}]$$

Unit of torque = $kg\ m^2 s^{-2}$

Work, Energy and Torque have same dimensional formula

15. $\text{Pressure} = \frac{\text{Force}}{\text{Area}}$

$$[P] = \frac{[MLT^{-2}]}{[L^2]}$$

$$[P] = [MLT^{-2}] \times [L^{-2}]$$

$$[P] = [ML^{-1}T^{-2}]$$

Unit of pressure $= \text{kg m}^{-1} s^{-2}$ or pascal(Pa)

16. $\text{Stress} = \frac{\text{Force}}{\text{Area}}$

$$[\text{stress}] = \frac{[MLT^{-2}]}{[L^2]}$$

$$[\text{stress}] = [ML^{-1}T^{-2}]$$

Unit of stress $= \text{kg m}^{-1} s^{-2}$

Pressure and Stress have same dimensional formula

17. $\text{Power} = \frac{\text{Work}}{\text{Time}}$

$$[P] = \frac{[M L^2 T^{-2}]}{[T]}$$

$$[P] = [M L^2 T^{-2}] \times [T^{-1}]$$

$$[P] = [M L^2 T^{-3}]$$

Unit of power $= \text{kg m}^2 s^{-3}$ or watt(W)

Physical quantities having no dimension and no unit

$$\text{Strain} = \frac{\textit{Change in dimension}}{\textit{Original dimension}} = \frac{[L]}{[L]} = [L^0]$$

$$\text{Relative Density} = \frac{\textit{Density of substance}}{\textit{Density of water}} = \frac{[ML^{-3}]}{[ML^{-3}]} = [L^0]$$

Physical quantities having units, but no dimension

Plane angle
Solid Angle
Angular Displacement

Find the dimensional formula of Gravitational constant using equation $F = \frac{Gm_1m_2}{r^2}$

$$G = \frac{Fr^2}{m_1m_2}$$

$$[G] = \frac{[F][r^2]}{[m_1]\,[m_2]}$$

$$[G] = \frac{[MLT^{-2}][L^2]}{[M][M]}$$

$$= [MLT^{-2}]\,[L^2][M^{-1}][M^{-1}]$$

$$[G] = [M^{-1}L^3T^{-2}]$$

Find the dimensional formula of Planck's constant using equation $\lambda = \frac{h}{mv}$

$$h = \lambda mv$$

$$[h] = [\lambda][m][v]$$

$$[h] = [L][M]\,[LT^{-1}]$$

$$[h] = [ML^2T^{-1}]$$

Find the dimensional formula of Coefficient of Viscosity using equation $F = \eta A\left(\frac{dv}{dx}\right)$

$$\eta = \frac{F}{A\left(\frac{dv}{dx}\right)}$$

$$\eta = \frac{Fdx}{A\,dv}$$

$$[\eta] = \frac{[MLT^{-2}]\ x\,[L]}{[L^2]\,[L\,T^{-1}]}$$

$$[\eta] = [MLT^{-2}] \times [L] \times [L^{-2}] \times [L^{-1}] \times [T]$$

$$[\eta] = [ML^{-1}T^{-1}]$$

Dimensional analysis and its applications

1) Checking the Dimensional Consistency(correctness) of Equations
2) Deducing Relation among the Physical Quantities

1) Checking the Dimensional Consistency(correctness) of Equations

The principle called the principle of homogeneity of dimensions is used to check the dimensional correctness of an equation.

The principle of homogeneity states that, for an equation to be correct, the dimesions of each terms on both sides of the equation must be the same.

The magnitudes of physical quantities may be added or subtracted only if they have the same dimensions.

If $X + Y = Z$
$[X] = [Y] = [Z]$

We cannot add 5m and 10kg
Velocity cannot be added to force.

eg1) Check the dimensional consistency(correctness) or homogeneity of the equation

$$s = ut + \frac{1}{2}at$$

s= displacement
u =initial velocity
a=acceleration
t=time

$$[s] = L$$
$$[ut] = LT^{-1} \times T = L$$
$$[\frac{1}{2}at] = LT^{-2} \times T = LT^{-1}$$

Since the dimensions of all terms of the equation are not same, this equation is wrong.

2) Check the dimensional consistency(correctness) or homogeneity of the equation

$$s = ut + \frac{1}{2}at^2$$

s= displacement
u =initial velocity
a=acceleration
t=time

$$[s] = L$$
$$[ut] = LT^{-1} \times T = L$$
$$[\frac{1}{2}at^2] = LT^{-2} \times T^2 = L$$

Since each term on both sides of equation has the same dimension, this equation is dimensionally correct

If an equation passes this consistency test it is not proved right.

3)Check the dimensional consistency(correctness) or homogeneity of the equation

$s = ut + \frac{2}{3}at^2$

$[s] = L$

$[ut] = LT^{-1} \times T$

$= L$

$[\frac{2}{3}at^2] = LT^{-2} \times T^2$

$= L$

s= displacement
u =initial velocity
a=acceleration
t=time

Since each term on both sides of equation has the same dimension, this equation is dimensionally correct.

Eventhough this equation is is dimensionally correct,it is not an exact equation.

A dimensionally correct equation need not be an exact (correct) equation, but a dimensionally wrong (incorrect) must be wrong.

4) Check the dimensional consistency(correctness) or homogeneity of the equation

$\frac{1}{2}mv^2 = mgh$

$[\frac{1}{2}mv^2] = M\ [LT^{-1}]^2$

$= ML^2T^{-2}$

m=mas of th body
v= velocity of body
g=acceleration due to gravity
h=height

$[mgh] = M\ LT^{-2}\ L$

$= ML^2T^{-2}$

The dimensions of LHS and RHS are the same and hence the equation is dimensionally correct.

5) Check the dimensional correctness of the equation $E = m\,c^2$

$[E] = ML^2T^{-2}$

$[m\,c^2] = M\ [LT^{-1}]^2$

$= ML^2T^{-2}$

E=energy
m=mass
c=velocity of light

The dimensions of LHS and RHS are the same and hence the equation is dimensionally correct.

6) In the given equation $v = x + at$, find the dimensions of x. (where v = velocity, a = acceleration, t = time)

$v = x + at$

$[v] = [x] = [at]$

$[x] = [v]$

$[x] = LT^{-1}$

7) In the given equation $x = a + bt + ct^2$, find the dimensions of a, b and c. (where x is in metres and t in seconds)

$x = a + bt + ct^2$

$[x] = [a] = [bt] = [ct^2]$

$[a] = [x]$

$[a] = L$

$[bt] = [x]$

$[b] \times T = L$

$[b] = \frac{L}{T}$

$[b] = LT^{-1}$

$[ct^2] = [x]$

$[c] \times T^2 = L$

$[c] = \frac{L}{T^2}$

$[c] = LT^{-2}$

8) The SI unit of energy is $J = kg\, m^2\, s^{-2}$; that of speed v is ms^{-1} and of acceleration a is ms^{-2}. Using dimensional arguments, find which of the formulae can be considered for kinetic energy (K)

(a) $K = m^2 v^3$

(b) $K = \frac{1}{2} mv^2$

(c) $K = ma$

(d) $K = \frac{3}{16} mv^2$

(e) $K = \frac{1}{2} mv^2 + ma$

(a) $K = m^2 v^3$

$ML^2T^{-2} = [M]^2 [LT^{-1}]^3$

$\neq M^2 L^3 T^{-3}$

(b) $K = \frac{1}{2} mv^2$

$ML^2T^{-2} = [M] [LT^{-1}]^2$

$= ML^2T^{-2}$

(c) $K = ma$

$ML^2T^{-2} = M[LT^{-2}]$

$\neq MLT^{-2}$

(d) $K = \frac{3}{16} mv^2$

$ML^2T^{-2} = [M] [LT^{-1}]^2$

$= ML^2T^{-2}$

(e) $K = \frac{1}{2} mv^2 + ma$

$[ML^2T^{-2}] = [M][LT^{-1}]^2 + [M][LT^{-2}]$

$\neq ML^2T^{-2} + MLT^{-2}$

Since dimensions of all terms are the same for Equations (b) and (d), these equations can be considered as the equation for kinetic energy.

9.The Van der waals equation of 'n' moles of a real gas is $(P+\frac{a}{V^2})(V-b)=nRT$. Where P is the pressure, V is the volume, T is absolute temperature, R is molar gas constant and a, b, c are Van der waal constants. Find the dimensional formula for a and b.

$$(P+\frac{a}{V^2})(V-b)=nRT.$$

By principle of homegeneity, the quantities with same dimensions can be added or subtracted.

$$[P]=[\frac{a}{V^2}]$$

$$[a]=[PV^2]$$

$$=ML^{-1}T^{-2} \text{ x } L^6$$

$$[a]=ML^5T^{-2}$$

$$[b]=[V]$$

$$[b]=L^3$$

2) Deducing Relation among the Physical Quantities

We can deduce relation of a physical quantity which depends upto three physical quantities.

1)Derive the equation for kinetic energy(E) of a body of mass m moving with velocity v.

$$E \alpha \; m^x \; v^y$$

$$E = k\, m^x v^y \longrightarrow (1)$$

Writing the dimensions on both sides,

$$M L^2 T^{-2} = M^x (LT^{-1})^y$$

$$M^1 L^2 T^{-2} = M^x L^y T^{-y}$$

equating the dimensions on both sides,

$$x=1$$
$$y=2$$

Substituting in eq (1)

$$E = k\, m^1 v^2$$

$$\boxed{E = k\, mv^2}$$

2) Suppose that the period of oscillation of the simple pendulum depends on its mass of the bob (m), length (l) and acceleration due to gravity (g). Derive the expression for its time period using method of dimensions.

$$T \propto m^x \; l^y \; g^z$$

$$T = k\, m^x\, l^y\, g^z \longrightarrow (1)$$

Writing the dimensions on both sides,

$$M^0L^0T^1 = M^x\; L^y\; (LT^{-2})^z$$

$$M^0L^0T^1 = M^x\; L^y\; L^z\; T^{-2z}$$

$$M^0L^0T^1 = M^x\; L^{y+z}\; T^{-2z}$$

equating the dimensions on both sides,

$$x = 0$$

$$y + z = 0$$

$$-2z = 1 \qquad z = \frac{-1}{2}$$

$$y + \frac{-1}{2} = 0 \qquad y = \frac{1}{2}$$

$$\boxed{T = k m^0\, l^{1/2} g^{-1/2}}$$

$$T = k\frac{l^{1/2}}{g^{1/2}}$$

$$T = k\frac{\sqrt{l}}{\sqrt{g}}$$

$$\boxed{T = k\sqrt{\frac{l}{g}}}$$

Limitations of Dimensional Analysis

1) Dimensional analysis check only the dimensional correctness of an equation, but not the exact correctness.
2) The dimensionless constants cannot be obtained by this method.
3) We cannot deduce a relation, if a physical quantity depends on more than three physical quantities.
4) The method cannot be considered to derive equations involving more than one term
5) A formula containing trigonometric , exponential and logarithmic function can not be derived from it.
6) It does not distinguish between the physical quantities having same dimensions.

Significant Figures

The result of measurement is a number that includes all digits in the number that are known reliable plus the first digit that is uncertain.

The reliable digits plus the first uncertain digit in a measurement are known as significant digits or significant figures.

If the period of oscillation of a simple pendulum is 1.62 s, the digits 1 and 6 are reliable and certain, while the digit 2 is uncertain

Rules for the determination of number of significant figures

A choice of change of different units does not change the number of significant digits or figures in a measurement.

For example, the length 2.308 cm has four significant figures.

But in different units, the same value can be written
0.02308 m - 4 significant figures.

23.08 mm - 4 significant figures.

23080 μm - 4 significant figures.

Rule1: All the non-zero digits are significant.

38 - 2

123 - 3

23.453 - 5

Rule2: All the zeros between two non-zero digits are significant, no matter where the decimal point is,

1204 - 4

30007 - 5

20.03 - 4

Rule3: If the number is less than one, the zero(s) on the right of decimal point but to the left of the first non-zero digit are not significant. The zero conventionally put on the left of decimal point for a number less than one is not significant.

0.032 - 2

0.004002 - 4

0.0132 - 3

Rule4:For a number without any decimal point the terminal or trailing zero(s) are not significant.

4200 - 2

23040 - 4

38100 - 3

Rule5: For a number with a decimal point, the trailing zero(s) are significant.

20.0 - 3

43.00 - 4

1203.0 - 5

Rule 6:The power of 10, in scientific notation is irrelevant to the determination of significant figures.

Now suppose we change units,

then 4.700 m = 470.0 cm = 4700 mm = 0.004700 km

All these measurements have four significant figures and a mere change of units cannot change the number of significant figures.

To remove such ambiguities in determining the number of significant figures, the best way is to report every measurement in scientific notation (in the power of 10).

$$\begin{aligned} &4.700 \text{ m} \\ &= 4.700 \times 10^2 \text{ cm} \\ &= 4.700 \times 10^3 \text{ mm} \\ &= 4.700 \times 10^{-3} \text{ km} \end{aligned}$$

All these numbers have 4 significant namaste figures.

Rounding off the Uncertain Digits

1) If the insignificant digit to be dropped is more than 5, the preceding digit is raised by 1

A number 2.746 rounded off to three significant figures is 2.75

Here the insignificant digit , 6 > 5 and hence 1 is added to the preceeding digit 4 .(4+1=5)

2) If the insignificant digit to be dropped less than 5, the preceding digit is left unchanged .

A number 2.743 rounded off to three significant figures is be 2.74.

Here the insignificant digit , 3< 5 and hence the preceeding number 4 does not change.

3) If the insignificant digit to be dropped is 5,

Case i) If the preceding digit is even, the insignificant digit is simply dropped.

A number 2.745 rounded off to three significant figures is 2.74.

Here the preceding digit 4 is even and hence 5 is simply dropped.

Case ii-) If the preceding digit is odd, the preceding digit is raised by 1.

A number 2.735 rounded off to three significant figures is 2.74

Here the preceding digit 3 ,is odd and hence 1 is added to It. (3+1=5)

Rules for Arithmetic Operations with Significant Figures

(1)In multiplication or division, the final result should retain as many significant figures as are there in the original number with the least significant figures.

Eg:If mass of an object is measured to be, 4.237 g (four significant figures) and its volume is measured to be 2.51cm3(3 significant figures), then find its density in appropriate significant figures.

$$\text{Density} = \frac{mass}{volume} = \frac{4.237\ g}{2.51 cm^3} = 1.688047$$

As per rule the final result should be rounded to 3 significant figures .

So the answer is 1.69 g/ cm^3

(2) In addition or subtraction, the final result should retain as many decimal places as are there in the number with the least decimal places.

Eg:Find the sum of the numbers 436.32 g, 227.2 g and 0.301 g to appropriate significant figures.

436.32 g +	(2 decimal places)
227.2 g +	(1 decimal place)
0.301 g	(3 decimal places)
663.821 g	

As per rule ,the final result should be rounded to 1 decimal place.

So the answer 663.8 g

Chapter 2
Motion in a Straight Line

The study of motion of objects along a straight line is also known as rectilinear motion .

Instantaneous Velocity and Speed

Instantaneous velocity

The average velocity tells us how fast an object has been moving over a given time interval but does not tell us how fast it moves at different instants of time during that interval. For this, we define instantaneous velocity or simply velocity v at an instant t.

The velocity at an instant is called instantaneous velocity and is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$v = \frac{dx}{dt}$$

$\frac{dx}{dt}$ is the differential coefficient of x with respect to t .It is the rate of change of position with respect to time.

Determining velocity from position-time graph. Velocity at t = 4 s is the slope of the tangent to the graph at that instant.

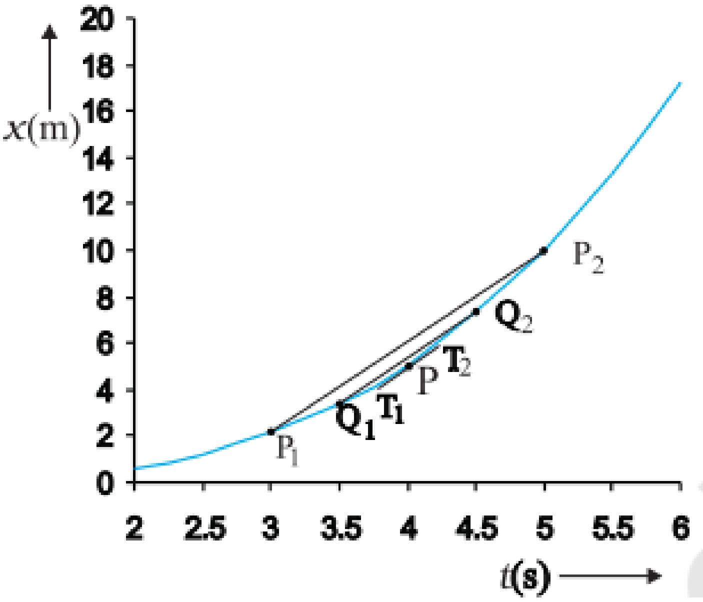


Table 2.1 Limiting value of $\frac{\Delta x}{\Delta t}$ at $t = 4$ s

Δt (s)	t_1 (s)	t_2 (s)	$x(t_1)$ (m)	$x(t_2)$ (m)	Δx (m)	$\Delta x / \Delta t$ ($m\ s^{-1}$)
2.0	3.0	5.0	2.16	10.0	7.84	3.92
1.0	3.5	4.5	3.43	7.29	3.86	3.86
0.5	3.75	4.25	4.21875	6.14125	1.9225	3.845
0.1	3.95	4.05	4.93039	5.31441	0.38402	3.8402
0.01	3.995	4.005	5.100824	5.139224	0.0384	3.8400

We see from Table 2.1 that as we decrease the value of Δt from 2.0 s to 0.010 s, the value of the average velocity approaches the limiting value 3.84 m s^{-1} which is the value of velocity at t = 4.0 s, i.e. the value of $\frac{dx}{dt}$ at t = 4.0 s.

Instantaneous speed

Instantaneous speed or simply speed is the magnitude of velocity. For example, a velocity of 24 m s^{-1} and a velocity of – 24 m s^{-1} — both have an associated speed of 24.0 m s^{-1}.

Example

The position of an object moving along x-axis is given by $x = a + bt^2$ where a = 8.5 m, b = 2.5 m s^{-2} and t is measured in seconds.

(a)What is its velocity at t = 0 s and t = 2.0 s.

(b) What is the average velocity between t = 2.0 s and t = 4.0 s ?

(a) $x = a + bt^2$

$$v = \frac{dx}{dt} = \frac{d}{dt}(a + bt^2) = 2bt$$

At t = 0 , v = 0

At t = 2 , v = 2 x 2.5 x 2 = 10 m s^{-1}

(b) $$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x_4 - x_2}{4-2}$$

$$= \frac{a+16b-a-4b}{2}$$

$$= \frac{12b}{2} = \frac{12 \times 2.5}{2} = 15 \text{ m s}^{-1}$$

Acceleration

Suppose the velocity itself is changing with time. In order to describe its effect on the motion of the particle, we require another physical quantity called acceleration. The rate of change of velocity of an object is called acceleration.

Average Acceleration

The average acceleration a over a time interval is defined as the change of velocity divided by the time interval .

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

- Unit of acceleration is ms^{-2} , [a] $= LT^{-2}$
- Acceleration is a vector quantity.
- If velocity is increasing with time, acceleration is +ve.
- If velocity is decreasing with time, acceleration is -ve.
- -ve acceleration is called retardation or deceleration.

Uniform acceleration

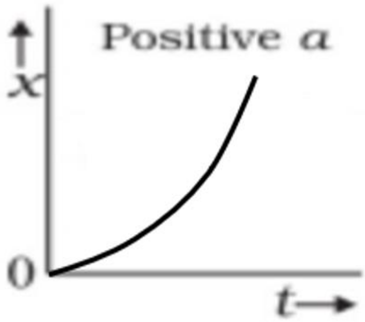
If the velocity of an object changes by equal amounts in equal intervals of time, it has uniform acceleration.

Instantaneous acceleration

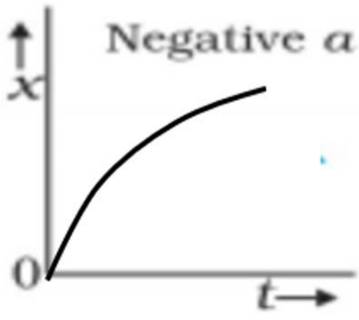
The acceleration of a particle at any instant of its motion is called instantaneous acceleration.

Position-time graph for motion with

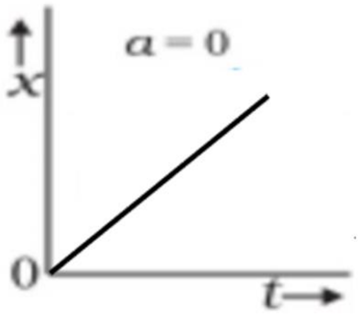
(a)positive acceleration



(b) negative acceleration

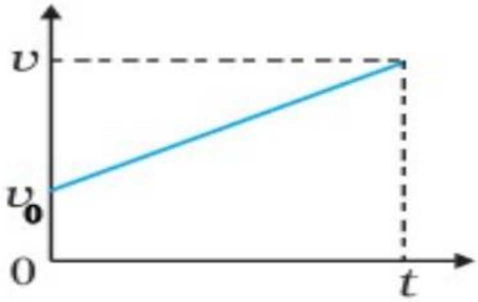


c)zero acceleration

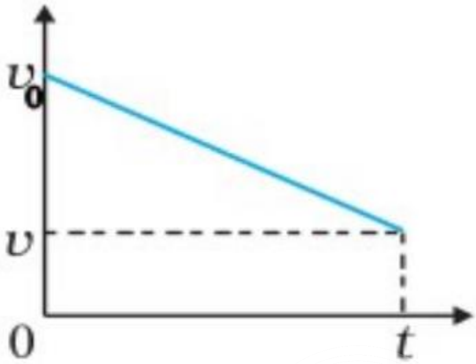


Velocity–time graph for motions with constant acceleration

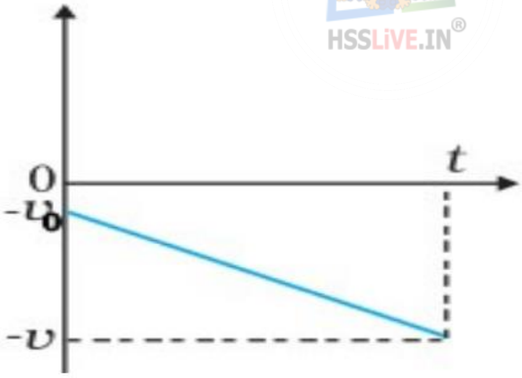
(a)Motion in positive direction with positive acceleration



(b) Motion in positive direction with negative acceleration

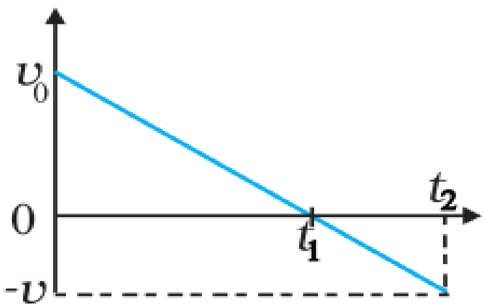


(c)Motion in negative direction with negative acceleration



(d)Motion of an object with negative acceleration that changes direction at time t_1.

(Between times 0 to t_1, its moves in positive x - direction and between t_1 and t_2 it moves in the opposite direction.)

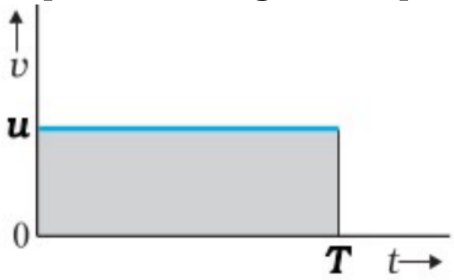


Importance of Velocity - time graph for a moving objct

An interesting feature of velocity - time graph for any moving objct is that **the area under the velocity - time graph is equal to the displacement of the particle.**

Proof for this statement :-

In uniform motion, velocity is the same at any instant of motion. Therefore, the velocity - time graph is a straight line parallel to the time axis.

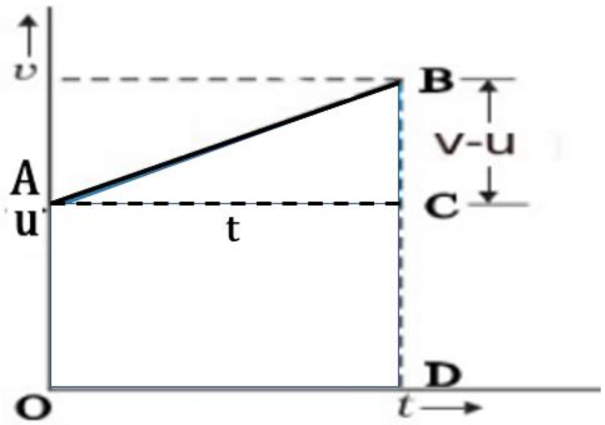


Area =uT = Displacement

i.e.,the area under the velocity - time graph is equal to the displacement of the particle.

Kinematic Equations for Uniformly Accelerated Motion

Consider a body moving with uniform acceleration . The velocity – time graph is as shown in figure



(1) Velocity – time relation

From the graph , acceleration = slope

$$a = \frac{BC}{AC}$$

$$a = \frac{v-u}{t}$$

$$v\text{-}u = at$$

$$\mathbf{v = u + at} \text{ ------------------- } \mathbf{(1)}$$

$$\text{or } (v = v_0 + at)$$

(2) Position-time relation

Displacement = Area under the graph

s = Area of ■ + Area of ▲

$s = ut + \frac{1}{2}(v-u)\,t$

But from equation (1)

$v - u = at$

$s = ut + \frac{1}{2}\,at \times t$

$$\mathbf{s = ut + \tfrac{1}{2}at^2} \text{ ------------------(2)}$$

or $(s = v_0 t + \frac{1}{2}at^2)$

(3)Position – velocity relation

Displacement = Average velocity x time

Average velocity $= \frac{v+u}{2}$

From equation (1) , $v - u = at$

$t = \frac{v-u}{a}$

Substituting the values,

$$s = \left(\frac{v+u}{2}\right)\left(\frac{v-u}{a}\right)$$

$$s = \left(\frac{v^2-u^2}{2a}\right)$$

$$v^2 - u^2 = 2as$$

$$\mathbf{v^2 = u^2 + 2as} \text{ ----------------(3)}$$

Or $(v^2 = v_0^2 + 2as)$

Stopping distance of vehicles

When brakes are applied to a moving vehicle, the distance it travels before stopping is called stopping distance.

$$v^2 = u^2 + 2as$$

$$0 = u^2 + 2as$$

$$-u^2 = 2as$$

$$s = \frac{-u^2}{2a}$$

Motion of an object under Free Fall

Free fall is a case of motion with uniform acceleration.

Since the acceleration due to gravity is always downward,

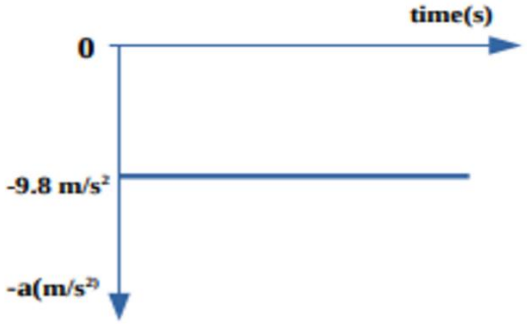
$$a = -g = -9.8\ ms^{-2}$$

The object is released from rest at y = 0. Therefore, u = 0

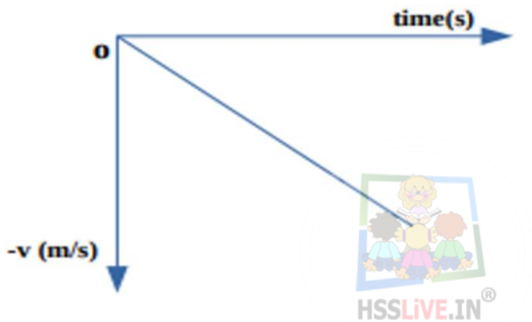
Then the equations of motion become

$$v = 0 - g\,t \quad = -9.8\,t$$
$$y = 0 - \tfrac{1}{2}\,g\,t^2 \quad = -4.9\,t^2$$
$$v^2 = 0 - 2\,g\,y \quad = -19.6\,y$$

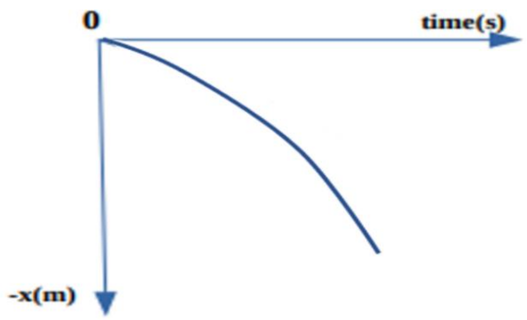
(a)Variation of acceleration with time



(b)Variation of velocity with time



(c)Variation of distance with time

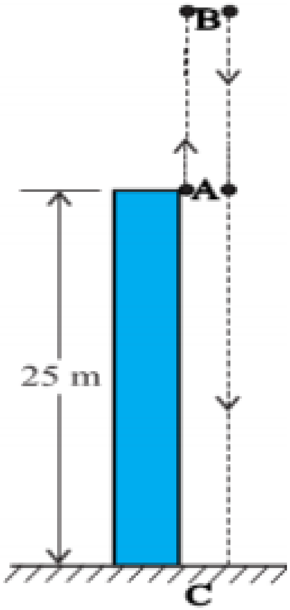


Example

A ball is thrown vertically upwards with a velocity of 20 m s^{-1} from the top of a multistorey building. The height of the point from where the ball is thrown is 25.0 m from the ground.

(a) How high will the ball rise ? and

(b) how long will it be before the ball hits the ground? Take g = 10 m s^{-2}



a) $u = 20$ m/s

$v = 0$

$a = -10 m/s^2$

$v^2 - u^2 = 2as$

$0 - 20^2 = 2 \times -10 \times s$

$-400 = -20\ s$

$s = -400/-20 = 20$ m

Total height = 20+ 25 = 45 m

(b) Total time = time fro upward motion + time for downward motion

For upward motion ,

$v = 0$

$u = 20$ m/s

$a = -10 m/s^2$

$v = u + at$

$0 = 20 + -10t$

$10\ t = 20 \qquad t = 20/10 = 2$ s

For downward motion,

$u = 0$

$s = -45$m

$a = -10 m/s^2$

$s = ut + ½ at^2$

$-45 = 0 - ½ \times 10 \times t^2$

$-45 = -5\ t^2 \qquad t^2 = 9, \quad t = 3$ s

Total time = 2 + 3 = 5s

Example

Galileo's law of odd numbers :"The distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity [namely, 1: 3: 5: 7......]." Prove it.

Answer Let us divide the time interval of motion of an object under free fall into many equal intervals τ and find out the distances traversed during successive intervals of time. Since initial velocity is zero, we have

$$y = - ½\ t^2$$

t	y	y in terms of y_o [=(- ½) g τ^2]	Distance traversed in successive intervals	Ratio of distances traversed
0	0	0		
τ	$-(1/2)\ g\ \tau^2$	y_o	y_o	1
2 τ	$-4(1/2)\ g\ \tau^2$	4 y_o	3 y_o	3
3 τ	$-9(1/2)\ g\ \tau^2$	9 y_o	5 y_o	5
4 τ	$-16(1/2)\ g\ \tau^2$	16 y_o	7 y_o	7
5 τ	$-25(1/2)\ g\ \tau^2$	25 y_o	9 y_o	9
6 τ	$-36(1/2)\ g\ \tau^2$	36 y_o	11 y_o	11

We find that the distances are in the simple ratio 1: 3: 5: 7: 9: 11... as shown in the last column. This law was established by Galileo Galilei.

Chapter 3
Motion in a Plane

Scalars and Vectors

A scalar quantity has only magnitude and no direction. It is specified completely by a single number, along with the proper unit.

Eg. distance ,mass , temperature, time .

A vector quantity has both magnitude and direction and obeys the triangle law of addition or the parallelogram law of addition. A vector is specified by giving its magnitude by a number and its direction.

Eg.displacement, velocity, acceleration and force.

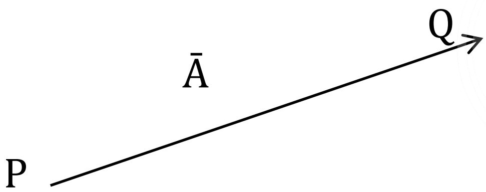
Representation of a Vector

A vector is representedby a bold letter say **A** or an arrow by an arrow placed over a letter, say $\bar{A}$.

The magnitude of a vector is called its absolute value, indicated by

$$|\bar{A}| = A$$

Graphically a vector is represented by a line segment with an arrow head.

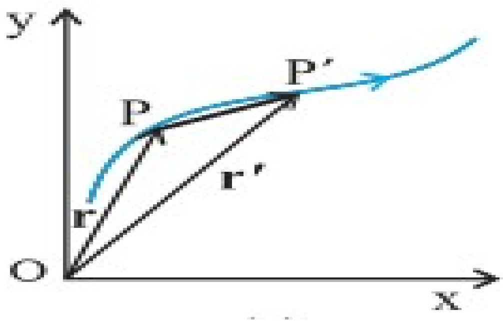


Q is the head of the vector

P is the tail of the vector

The length of line segment gives the magnitude of the vector and arrow mark gives its direction.

Position and Displacement Vectors

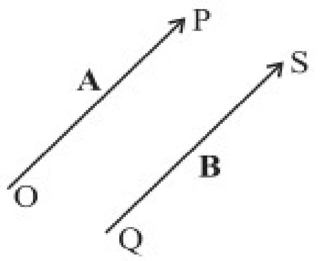


Let P and P′ be the positions of the object at time t and t′ , respectively . OP is the position vector of the object at time t. OP = r.
OP’ is the position vector of the object at time t’. OP’= r’
If the object moves from P to P′ , the vector PP′ is called the displacement vector.

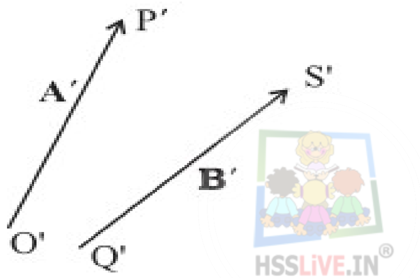
Displacement vector is the straight line joining the initial and final positions and does not depend on the actual path undertaken by the object between the two positions.

Equality of Vectors

Two vectors A and B are said to be equal if, and only if, they have the same magnitude and the same direction.
(a) Two equal vectors **A** and **B**.



(b) Two vectors **A**′ and **B**′ are unequal eventhough they are of same length



Multiplication of Vectors by Real Numbers

- Multiplying a vector $\bar{A}$ with a positive number λ gives a vector whose magnitude is changed by the factor λ but direction is the same as that of $\bar{A}$

$$\lambda \times \bar{A} = \lambda \bar{A}, \qquad \text{if } \lambda > 0$$

For example, if $\bar{A}$ is multiplied by 2, the resultant vector 2 $\bar{A}$ is in the same direction as $\bar{A}$ and has a magnitude twice of $|\bar{A}|$

- Multiplying a vector $\bar{A}$ by a negative number λ gives a vector $\lambda\bar{A}$ whose direction is opposite to the direction of $\bar{A}$ and whose magnitude is
$-\lambda$ times $|\bar{A}|$.

For example, multiplying a given vector A by negative numbers, say –1 and –1.5, gives vectors as

Null vector or a Zero vector

A Null vector or a Zero vector is a vector having zero magnitude and is represented by **O** or $\bar{O}$. The result of adding two equal and opposite vectors will be a Zero vector

Eg: When a body returns to its initial position its displacement will be a zero vector.

The main properties of $\bar{O}$ are :

$$\bar{A} + \bar{O} = \bar{A}$$
$$\lambda\,\bar{O} = \bar{O}$$
$$\bar{O}\,\bar{A} = \bar{O}$$

Unit vectors

A unit vector is a vector of unit magnitude and points in a particular direction.

It has no dimension and unit. It is used to specify a direction only.

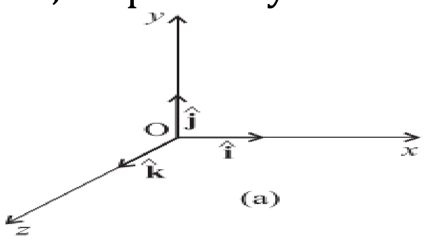
If we multiply a unit vector, say n by a scalar, the result is a vector.

In general, a vector A can be written as

$$\bar{A} = |A|$$
$$\hat{\mathbf{A}} = \frac{\bar{\mathbf{A}}}{|\mathbf{A}|}$$

where $\hat{A}$ is the unit vector along $\bar{A}$

Unit vectors along the x-, y- and z-axes of a rectangular coordinate system are denoted by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, respectively.

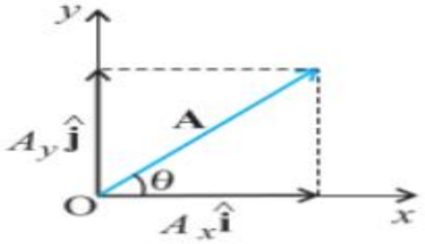


Since these are unit vectors, we have $|\,\hat{\imath}\,| = |\,\hat{\jmath}\,| = |\,\hat{k}\,| = 1$

These unit vectors are perpendicular to each other and are called orthogonal unit vectors

Resolution of a vector

We can now resolve a vector A in terms of component vectors that lie along unit vectors $\hat{i}$ and $\hat{j}$.



$$\bar{A} = A_x\,\hat{i} + A_y\,\hat{j}$$

The quantities A_x and A_y are called x-, and y- components of the vector A.
Note that A_x and A_y are not vectors, but $A_x\hat{i}$ and $A_y\hat{j}$ are vectors.

Using simple trigonometry, we can express A_x and A_y in terms of the magnitude of A and the angle θ it makes with the x-axis :

$$\cos\theta = \frac{A_x}{A}$$

$$A_x = A\cos\theta$$

$$\sin\theta = \frac{A_y}{A}$$

$$A_y = A\sin\theta$$

$$A_x^2 + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta = A^2$$

$$A = \sqrt{A_x^2 + A_y^2}$$

And $\tan\theta = \frac{A_y}{A_x}$, $\theta = \tan^{-1}\frac{A_y}{A_x}$

In general, if $\bar{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_y\,\hat{k}$

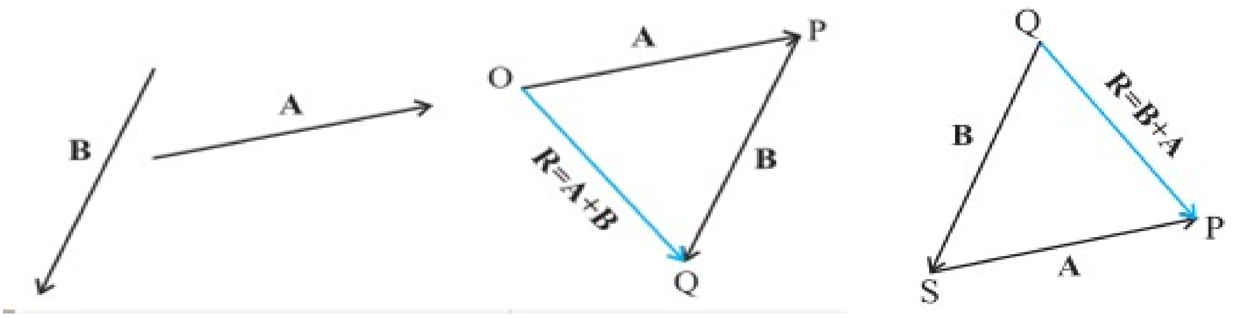
The magnitude of vector A is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Addition and Subtraction of Vectors — Graphical Method

Triangle law of vector addition

If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order ,then their resultant is given by the third side of the triangle taken in reverse order.



This graphical method is called the head-to-tail method.
If we find the resultant of B + A , the same vector R is obtained.

- Thus, vector addition is commutative:

$$A + B = B + A$$

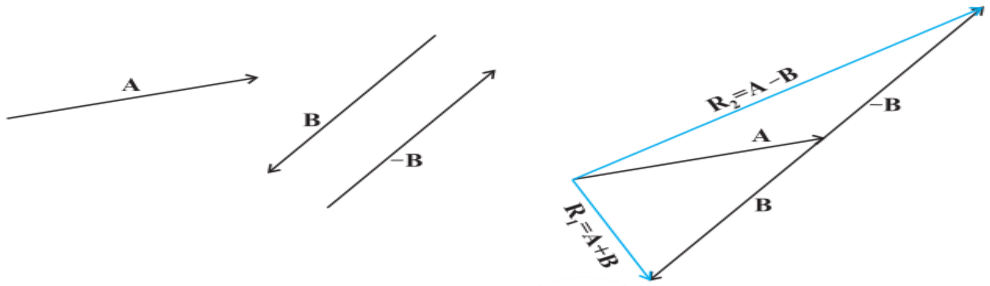
- The addition of vectors also obeys the associative law

$$(A + B) + C = A + (B + C)$$

Subtraction of vectors

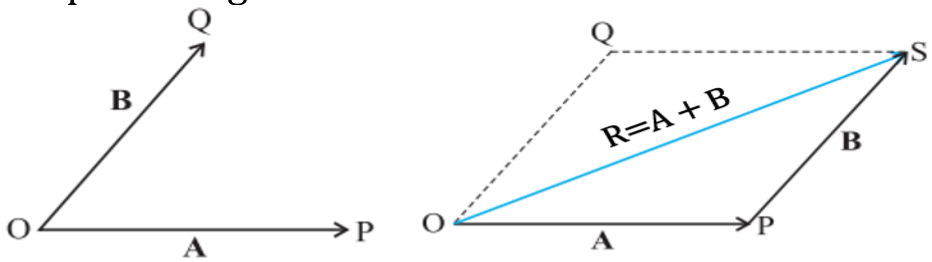
Subtraction of vectors can be defined in terms of addition of vectors. We define the difference of two vectors A and B as the sum of two vectors A and –B :

$$A - B = A + (-B)$$



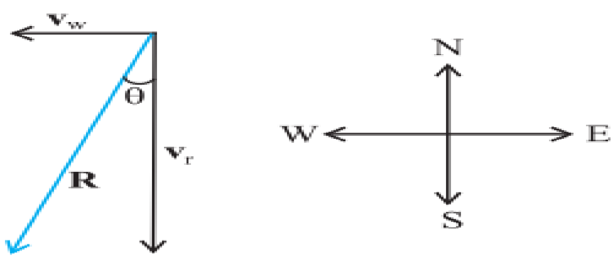
Parallelogram law of vector addition

If two vectors are represented in magnitude and direction by the adjacent sides of a parallelogram ,then their resultant is given by the diagonal of the parallelogram.



Example

Rain is falling vertically with a speed of 35 m s $^{-1}$. Winds starts blowing after sometime with a speed of 12 m s $^{-1}$ in east to west direction. In which direction
should a boy waiting at a bus stop hold his umbrella ?



$$R = \sqrt{v_r^2 + v_w^2} = \sqrt{35^2 + 12^2}\ \text{m s}^{-1} = 37\ \text{m s}^{-1}$$

The direction θ that R makes with the vertical is given by

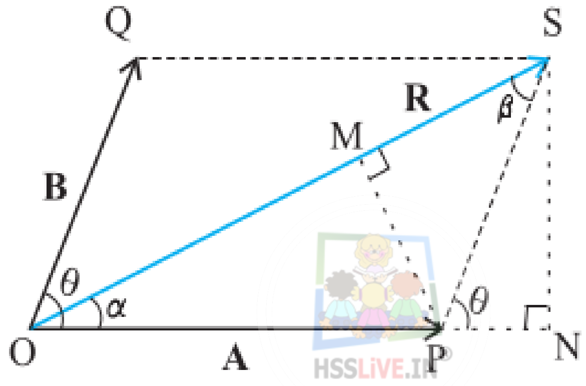
$$\tan\theta = \frac{v_w}{v_r} = \frac{12}{35} = 0.343$$

$$\theta = \tan^{-1}(0.343) = 19^\circ$$

Therefore, the boy should hold his umbrella in the vertical plane at an angle of about 19° with the vertical towards the east.

Vector Addition – Analytical Method

Consider two vectors **A** and **B** in x-y plane



SN is normal to OP and PM is normal to OS.

ΔSNP, $\cos\theta = PN / PS$ $\quad$ $\sin\theta = SN / PS$

$\cos\theta = PN / B$ $\quad$ $\sin\theta = SN / B$

$PN = B\cos\theta$ $\quad$ $SN = B\sin\theta$

From the geometry of the figure,

$$OS^2 = ON^2 + SN^2$$

$$\text{but } ON = OP + PN$$

$$= A + B\cos\theta$$

$$SN = B\sin\theta$$

$$OS^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

$$R^2 = A^2 + 2AB\cos\theta + B^2\cos^2\theta + B^2\sin^2\theta$$

$$R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\mathbf{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

This Equation gives the magnitude of the resultant of vectors A and B.

From figure,

$$\frac{R}{\sin\theta} = \frac{B}{\sin\alpha}$$

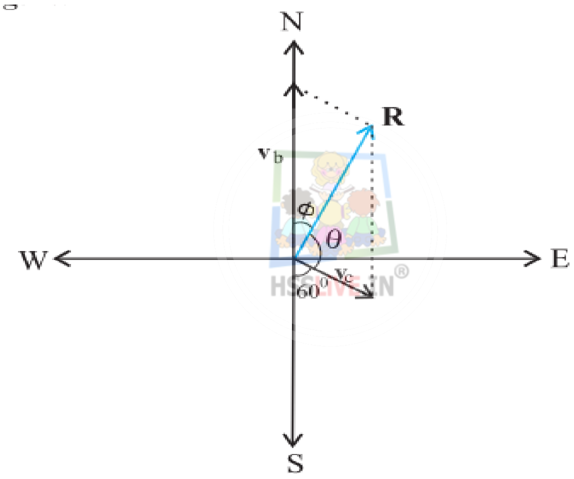
$$\sin\alpha = \frac{B}{R}\sin\theta$$

$$\tan\alpha = \frac{SN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta}$$

These Equations gives the direction of the resultant of vectors A and B.

Example

A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat



$$R = \sqrt{v_b^2 + v_c^2 + 2v_b v_c \cos 120^\circ}$$

$$= \sqrt{25^2 + 10^2 + 2\times25\times10(-1/2)} \cong 22 \text{ km/h}$$

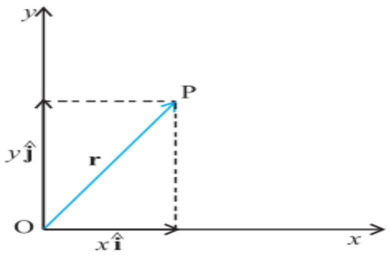
To obtain the direction, we apply the Law of sines

$$\frac{R}{\sin\theta} = \frac{v_c}{\sin\phi} \quad \text{or, } \sin\phi = \frac{v_c}{R}\sin\theta$$

$$= \frac{10\times\sin 120^\circ}{21.8} = \frac{10\sqrt{3}}{2\times21.8} \cong 0.397$$

$$\phi \cong 23.4^\circ$$

Motion in a Plane



Position Vector

The position vector r of a particle P at time t

$$\mathbf{r} = x\hat{i} + y\hat{j}$$

The position vector r of a particle P at time t'

$$\mathbf{r'} = x'\hat{i} + y'\hat{j}$$

Displacement vector

$$\Delta\mathbf{r} = \mathbf{r'} - \mathbf{r}$$

$$\Delta\mathbf{r} = (x'\hat{i} + y'\hat{j}) - (x\hat{i} + y\hat{j})$$

$$\Delta\mathbf{r} = (x' - x)\hat{i} + (y' - y)\hat{j}$$

$$\Delta\mathbf{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

Velocity vector

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t}$$

$$\mathbf{v} = \frac{\Delta x\,\hat{i} + \Delta y\,\hat{j}}{\Delta t}$$

$$\mathbf{v} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j}$$

$$\mathbf{v} = v_x\,\hat{i} + v_y\,\hat{j}$$

Instantaneous velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{v} = v_x\,\hat{i} + v_y\,\hat{j} \quad \text{where}$$

$$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}$$

The magnitude of $\mathbf{v}$ is then

$$v = \sqrt{v_x^2 + v_y^2}$$

and the direction of $\mathbf{v}$ is given by the angle θ :

$$\tan\theta = \frac{v_y}{v_x}, \quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Acceleration

$$\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta\left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}\right)}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}}$$

$$a = a_x\,\hat{i} + a_y\,\hat{j}$$

Instantaneous Acceleration

$$a = \frac{dv}{dt}$$

Or, $\mathbf{a} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}$

▶ ***Example 4.4*** The position of a particle is given by

$$\mathbf{r} = 3.0t\,\hat{\mathbf{i}} + 2.0t^2\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}$$

where t is in seconds and the coefficients have the proper units for $\mathbf{r}$ to be in metres. (a) Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ of the particle. (b) Find the magnitude and direction of $\mathbf{v}(t)$ at $t = 1.0$ s.

Answer

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(3.0\,t\,\hat{\mathbf{i}} + 2.0t^2\,\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}\right)$$

$$= 3.0\hat{\mathbf{i}} + 4.0t\,\hat{\mathbf{j}}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = +4.0\,\hat{\mathbf{j}}$$

$a = 4.0$ m s^{-2} along y- direction

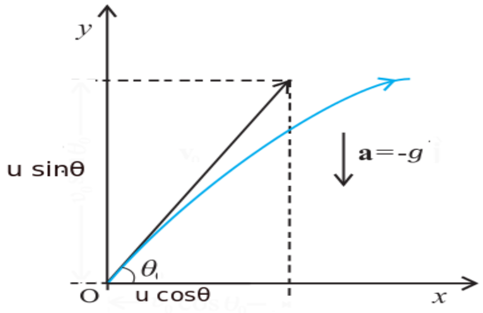
At $t = 1.0$ s, $\mathbf{v} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$

It's magnitude is $v = \sqrt{3^2 + 4^2} = 5.0$ m s^{-1}

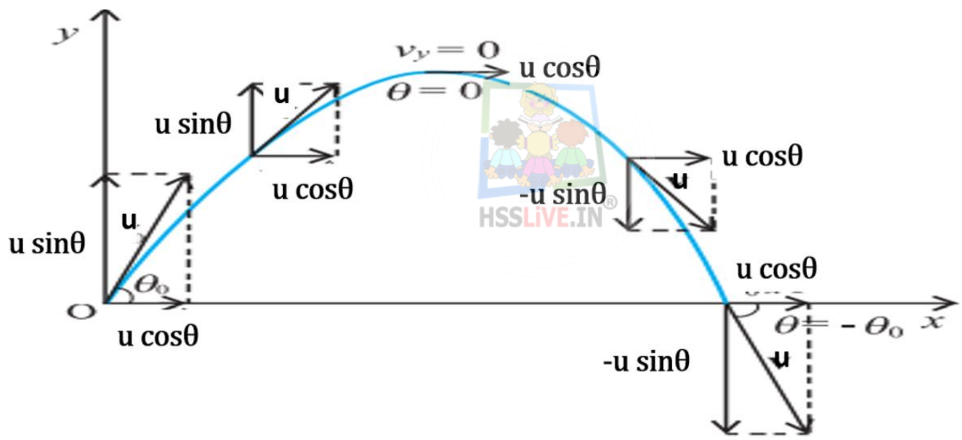
and direction is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) \cong 53^\circ \text{ with } x\text{-axis.}$$

Projectile Motion



- An object that is in flight after being thrown or projected is called a projectile.
- The path (trajectory)of a projectile is a parabola.
- The components of initial velocity u are **u cos θ** along horizontal direction and **u sin θ** along vertical direction.
- The x-component of velocity(**u cos θ**) remains constant throughout the motion and hence there is no acceleration in horizontal direction,i.e., $a_x = 0$
- The y- component of velocity (**u sin θ**) changes throughout the motion. At the point of maximum height, **u sin θ** = 0. There is acceleration in horizontal direction, $a_y = -g$



Equation of path of a projectile

Displacement of the projectile after a time t

$$x = u\cos\theta\, t$$

$$t = \frac{x}{u\cos\theta}$$

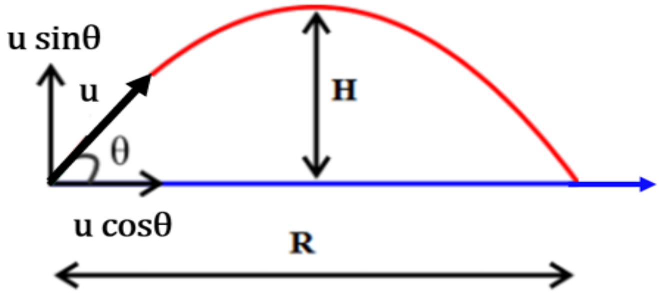
$$y = u\sin\theta\, t - \frac{1}{2} g\, t^2$$

$$y = u\sin\theta \left(\frac{x}{u\cos\theta}\right) - \frac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2$$

$$y = \tan\theta\, x - \frac{g}{2\, u^2\cos^2\theta}\, x^2$$

This equation is of the form $y = a\,x + b\,x^2$, in which a and b are constants. This is the equation of a parabola, i.e. **the path of the projectile is a parabola** .

Time of Flight of a projectile (T)



The total time T during which the projectile is in flight is called Time of Flight, T.

Consider the motion in vertical direction,

$$s = ut + \frac{1}{2}at^2$$

$s=0$,
$u = u \sin\theta$,
$a = -g$,
$t = T$

$$0 = u \sin\theta \, T - \frac{1}{2} g T^2$$

$$u \sin\theta \, T = \frac{1}{2} g T^2$$

$$\mathbf{T = \frac{2 u \sin\theta}{g}}$$

Horizontal range of a projectile (R)

The horizontal distance travelled by a projectile during its time of flight is called the horizontal range, R.

Horizontal range = Horizontal component of velocity x Time of flight

$$R = u\cos\theta \times \frac{2 u \sin\theta}{g}$$

$$R = \frac{u^2 \times 2 \sin\theta \cos\theta}{g}$$

$$\mathbf{R = \frac{u^2 \sin 2\theta}{g}}$$

R is maximum when sin2θ is maximum, i.e., when θ = 45 ^{0}.

$$\mathbf{R_{max} = \frac{u^2}{g}}$$

Show that for a given velocity of projection range will be same for angles θ and (90-θ)

For angle θ, $R = \frac{u^2 \sin 2\theta}{g}$

For angle (90 -θ), $R = \frac{u^2 \sin 2(90-\theta)}{g}$

$$R = \frac{u^2 \sin (180-2\theta)}{g}$$

$\sin (180 - 2\theta) = \sin 2\theta$

$$R = \frac{u^2 \sin 2\theta}{g}$$

for given velocity of projection range will be same for angles θ and (90-θ)

Maximum height of a projectile (H)

It is the maximum height reached by the projectile.

Consider the motion in vertical direction to the highest point

$$v^2 - u^2 = 2as$$

$u = u \sin\theta$,

$v = 0$,

$a = -g$,

$s = H$

$$0 - u^2 \sin^2\theta = -2 g H$$

$$H = \frac{u^2 \sin^2\theta}{2g}$$

Example

A cricket ball is thrown at a speed of 28 m s $^{-1}$ in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball

returns to the same level.

(a) $H = \frac{u^2 \sin^2\theta}{2g}$

$$H = \frac{28^2 \sin^2 30}{2 \times 9.8}$$

$$H = 10 \text{ m}$$

(b) $T = \frac{2\,u \sin\theta}{g}$

$T = \frac{2 \times 28 \sin 30}{9.8}$

$T = 2.9$ s

(c) $R = \frac{u^2 \sin 2\theta}{g}$

$R = \frac{28^2 \sin 60}{9.8}$

$R = 69$ m

Uniform Circular Motion

When an object follows a circular path at a constant speed, the motion of the object is called **uniform circular motion**. The word “uniform” refers to the speed, which is uniform (constant) throughout the motion.

Period

The time taken by an object to make one revolution is known as its time period T

Frequency

The number of revolutions made in one second is called its frequency.

$$\upsilon = \frac{1}{T}$$

unit - hertz (Hz)

Angular velocity (ω)

angular velocity is the time rate of change of angular displacement

$$\omega = \frac{\Delta\theta}{\Delta t}$$

In the limit Δt tends to zero

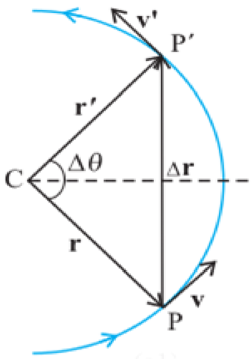
$$\omega = \frac{d\theta}{dt}$$

Unit is rad/s

During the time period T ,the angular displacement is 2π radian

$$\omega = \frac{2\pi}{T} \quad \text{or} \quad \omega = 2\pi\upsilon$$

Relation connecting angular velocity and linear velocity



As the object moves from P to P′ in time Δt . Δθ is called angular displacement and Δ r is the linear diplacement

$$\text{angle} = \frac{\text{arc}}{\text{radius}}$$

$$\Delta\,\theta = \frac{\Delta\,r}{r}$$

$$\Delta\,r = r\ \Delta\,\theta$$

$$\frac{\Delta\,r}{\Delta t} = r\,\frac{\Delta\,\theta}{\Delta t}$$

$$\mathbf{v = r\,\omega}$$

Angular Acceleration

The rate of change of angular velocity is called angular acceleration.

$$\alpha\ = \frac{d\,\boldsymbol{\omega}}{dt}$$

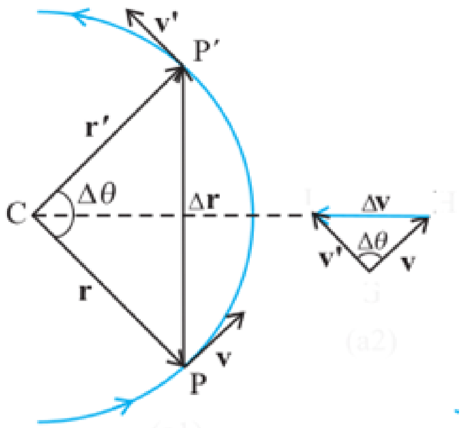
$$\text{But}\quad \omega = \frac{d\,\theta}{dt}$$

$$\alpha = \frac{d}{dt}\left(\frac{d\,\theta}{dt}\right)$$

$$\boldsymbol{\alpha = \frac{d^2\,\theta}{dt}}$$

Centripetal acceleration

A body in uniform circular motion experiences an acceleration , which is directed towards the centre along its radius .This is s called centripetal acceleration .



Let r and r′ be the position vectors and v and v′ the velocities of the object when it is at point P and P′

$$\frac{\Delta v}{v} = \frac{\Delta r}{r}$$

$$\Delta v = \frac{v \Delta r}{r}$$

$$\frac{\Delta v}{\Delta t} = \frac{v \Delta r}{r \Delta t}$$

$$a = \frac{v}{r} \times r$$

$$a = \frac{v^2}{r}$$

If R is the radius of circular path, then centripetal acceleration .

$$a_c = \frac{v^2}{R}$$

Centripetal acceleration can also be expressed as

$$v = R\omega$$

$$a_c = \frac{v^2}{R}$$

$$a_c = \frac{R^2 \omega^2}{R}$$

$$a_c = \omega^2 R$$

$$v = R\omega$$

$$R = v/\omega$$

$$a_c = \frac{v^2}{R}$$

$$a_c = \frac{v^2}{(v/\omega)}$$

$$a_c = v\omega$$

Example

An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s.
(a) What is the angular speed, and the linear speed of the motion?
(b) Is the acceleration vector a constant vector ? What is its magnitude ?

Period, $T = \frac{100}{7}$ s

(a) The angular speed ω is given by

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{100}{7}} = \frac{2\pi \times 7}{100} = 0.44 \text{ rad/s}$$

The linear speed , $v = \omega R = 0.44 \times 0.12 = 5.3 \times 10^{-2}$ m s^{-1}

(b) The direction of velocity v is along the tangent to the circle at every point. The acceleration is directed towards the centre of the circle. Since this direction changes continuously, acceleration here is not a constant vector.

$$a = \omega^2 R = (0.44)^2 \times 0.12 = 2.3 \times 10^{-2} \text{ m s}^{-2}$$

Chapter 4
Laws of Motion

Galileo's Law of Inertia

If the net external force is zero, a body at rest continues to remain at rest and a body in motion continues to move with a uniform velocity.
This property of the body is called inertia.Inertia means 'resistance to change'.

Suppose a person is standing in a stationary bus and the driver starts the bus suddenly. He gets thrown backward with a jerk. This is due to his inertia of rest.

Similarly if a person is standing in a moving bus and if the bus suddenly stops he is thrown forward. This is due to his inertia of motion.

Newton's Laws of Motion

Newton built on Galileo's ideas and laid the foundation of mechanics in terms of three laws of motion. Galileo's law of inertia was his starting point on which he formulated as the First Law of motion.

Newton's First Law of Motion (Law of inertia)

Every body continues to be in its state of rest or of uniform motion in a straight line unless compelled by some external force to change that state.

The state of rest or uniform linear motion both imply zero acceleration. If the net external force on a body is zero, its acceleration is zero. Acceleration can be non zero only if there is a net external force on the body.

Momentum

Momentum, P of a body is defined to be the product of its mass m and velocity v, and is denoted by p.

$$p = m\,v$$

Momentum is a vector quantity.

Unit = kgm/s

$$[p] = ML\,T^{-1}$$

- Suppose a light-weight vehicle car) and a heavy weight vehicle are parked on a horizontal road. A greater force is needed to push the truck than the car to bring them to the sam e speed in same time. Similarly, a greater opposing force is needed to stop a heavy body than a light body in the same time, if they are moving with the same speed.
- Speed is another important parameter to consider. A bullet fired by a gun can easily pierce human tissue before it stops, resulting in casualty. The same bullet fired with moderate speed will not cause much damage.Thus for a given mass, the greater the speed, the greater is the opposing force needed to stop the body in a certain time.

Newton's Second Law f Motion

The rate of change of momentum of a body is directly proportional to the applied force and takes place in the direction in which the force acts.

$$\mathbf{F} \propto \frac{\Delta \mathbf{p}}{\Delta \mathbf{t}}$$

$$F = k\frac{\Delta p}{\Delta t}$$

For simplicity we choose k=1

In the limit $\Delta t \to 0$

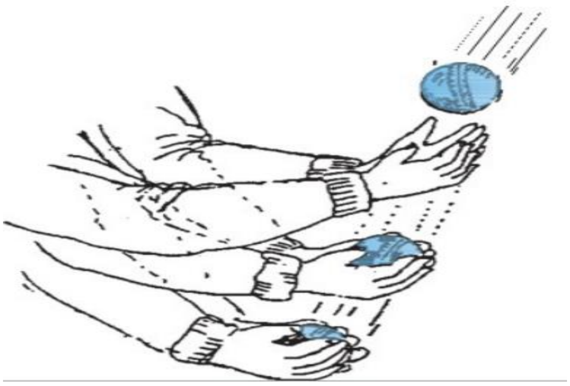
$$\mathbf{F} = \frac{\mathbf{dp}}{\mathbf{dt}}$$

Why a cricketer draws his hands backwards during a catch?

By Newton's second law of motion ,

$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta \mathbf{t}}$$

When he draws his hands backwards, the time interval (Δt) to stop the ball increases . Then force decreases and it does not hurt his hands. Force not only depends on the change in momentum but also on how fast the change is brought about.



Derivation of Equation of force from Newton's second law of motion

By Newton's second law of motion ,

$$F = \frac{dp}{dt}$$

For a body of fixed mass m, p=mv

$$F = \frac{d}{dt}mv$$

$$F = m\frac{dv}{dt}$$

$$F = ma$$

Force is a vector quantity

Unit of force is $kgms^{-2}$ or newton (N)

Definition of newton

$$F = m\,a$$

If $m=1\ kg$, $a=1 m\,s^{-2}$

$$F=1kg\ x\ 1\ ms^{-2}$$
$$F=1N$$

One newton is that which causes an acceleration of m s^{-2} to a mass of 1kg

Important points about second law

1.The second Law is consistent with the first law.

From Newton's second law,

$$F = ma$$

$$\text{If } F = 0,\quad ma = 0$$

$$(\text{since } m \neq 0)$$

$$a = 0$$

Zero acceleration implies the state of rest or uniform linear motion . i.e, when there is no external force , the body will remain in its state of rest or of uniform motion in a straight line. This is Newtons first law of motion.

2. The second law of motion is a vector law.

$$F_x = \frac{dP_x}{dt} = m\frac{dv_x}{dt} = ma_x$$

$$F_y = \frac{dP_y}{dt} = m\frac{dv_y}{dt} = ma_y$$

$$F_z = \frac{dP_z}{dt} = m\frac{dv_z}{dt} = ma_z$$

3. If a force makes some angle with velocity, the force changes only the component of velocity along the direction of force.

4. For a system of particles we can write

$$F_{net} = ma$$

F_{net} refers to the total external force on the system and a refers to the acceleration of the system as a whole.

5. In equation F=ma

Acceleration at any instant is determined by the force at that instant , not by any history of the motion of the particle.

Example

A bullet of mass 0.04 kg moving with a speed of 90 m/s enters a heavy wooden block and is stopped after a distance of 60 cm. What is the average resistive force exerted by the block on the bullet?

m=0.04 kg

u=90 m/s

v=0

s=60cm=0.6 m

The retardation 'a' of the bullet is assumed to be constant.

$$v^2 - u^2 = 2as$$

$$0 - 90^2 = 2 \text{x a x} 0.6$$

$$a = \frac{-90^2}{2 \text{ x } 0.6}$$

$$a = -6750 \text{ m}/s^2$$

The retarding force, F= ma

$$F = 0.04 \text{ x} - 6750$$

$$F = -270 \text{ N}$$

The negative sign shows that the force is resistive or retarding.

Example

The motion of a particle of mass m is described by $y = ut + \frac{1}{2}gt^2$. Find the force acting on the particle.

$$y = ut + \frac{1}{2}gt^2$$

Velocity , $v = \frac{dy}{dt}$

$$v = \frac{d}{dt}(ut + \frac{1}{2}gt^2)$$

$$v = u\frac{dt}{dt} + \frac{1}{2}g\frac{d}{dt}t^2)$$

$$v = u + \frac{1}{2}g \; x \; 2t$$

$$\frac{d}{dt}t^n = nt^{n-1}$$

$$\frac{d}{dt}t^2 = 2t^{2-1} = 2t$$

$$v = u + gt$$

Acceleration, $a = \frac{dv}{dt}$

$$a = \frac{d}{dt}(u + gt)$$

$$a = \frac{du}{dt} + g\frac{dt}{dt})$$

$$a = g$$

Force, $F = ma$

$$F = mg$$

Impulse (I)

There are some situations where a large force acts for a very short duration producing a finite change in momentum of the body. For example, when a ball hits a wall and bounces back, the force on the ball by the wall acts for a very short time when the two are in contact, yet the force is large enough to reverse the momentum of the ball.

Impulse is the the product of force and time duration, which is the change in momentum of the body.

Impulse = Force × time duration

$$I = F \; x \; t$$

Unit = $kg \; m \; s^{-1}$

$[I] = M \; L \; T^{-1}$

Impulsive force.

A large force acting for a short time to produce a finite change in momentum is called an impulsive force.

Impulse momentum Principle

Impulse is equal to the change in momentum of the body.

By Newton's second law of motion,

$$F=\frac{dp}{dt}$$

$$F \times dt = dp$$

$$I = dp$$

Impulse = change in momentum

Example

A batsman hits back a ball straight in the direction of the bowler without changing its initial speed of 12 m s^{-1} . If the mass of the ball is 0.15 kg, determine the impulse imparted to the ball.

Impulse = change of momentum

Change in momentum = final momentum – initial momentum

Change in momentum = $0.15 \times 12-(0.15\times-12)$

Impulse = 3.6 N s

Newton's Third Law of Motion

To every action, there is always an equal and opposite reaction.

Important points about Third law

1. Forces always occur in pairs. Force on a body A by B is equal and opposite to the force on the body B by A.

2. There is no cause- effect relation implied in the third law. The force on A by B and the force on B by A act at the same instant. By the same reasoning, any one of them may be called action and the other reaction.

3. Action and reaction forces act on different bodies, not on the same body. So they do not cancel each other , eventhough they are equal and opposite.

According to the third law,

$F_{AB} = -F_{BA}$ (force on A by B) = – (force on B by A)

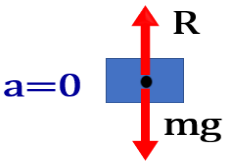
4. However, if you are considering the system of two bodies as a whole, F_{AB} and F_{BA} are internal forces of the system (A + B). They add up to give a null force.

Weight of a body in a lift

Weight of a body is the normal reaction exerted by the surface in contact with that body.

Case:1

When lift is at rest(or moving up or down with uniform velocity)

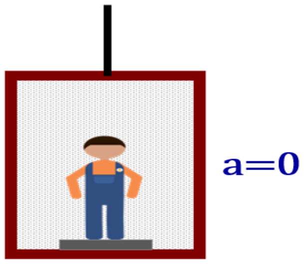


$F_{net} = ma$

$R\text{-}mg = m \times 0$

$R\text{-}mg = 0$

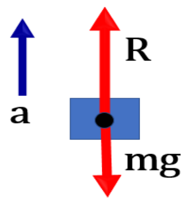
$R = mg$



No change in weight of the body.

Case 2

When lift is moving up with an acceleration 'a'

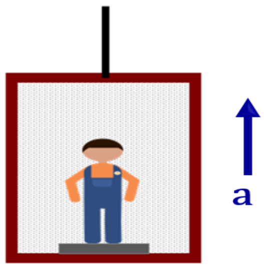


$F_{net} = ma$

$R - mg = ma$

$R = mg + ma$

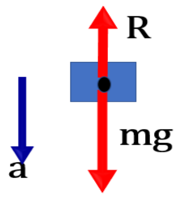
$R = m(g + a)$



The weight of the body increases

Case 3

When lift is moving down with an acceleration 'a'

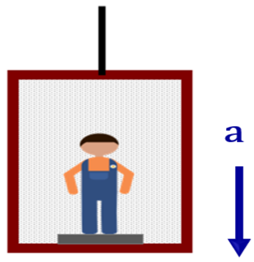


$F_{net} = ma$

$mg - R = ma$

$R = mg - ma$

$R = m(g - a)$



The weight of the body decreases

Case 4

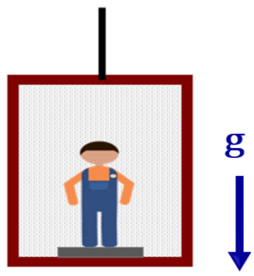
When the lift mechanism failed and it moves down freely under gravity.

$R = mg - ma$

$a = g$

$R = m(g - g)$

$R = 0$



The body experiences weightlessness.

Example

A man of mass 70 kg stands on a weighing scale in a lift which is moving,
(a) upwards with a uniform speed of 10 m s^{-1}
(b) downwards with a uniform acceleration of 5 m s^{-2}
(c) upwards with a uniform acceleration of 5 m s^{-2}
What would be the readings on the scale in each case?
(d) What would be the reading if the lift mechanism failed and it falls down freely under gravity? Take g=10 **m s^{-2}**

(a)When lift moves with uniform speed , a=0

R= mg = 70 x 10=700 N

Reading= 700/ 10= 70 kg

(b)Acceleration a = 5m s^{-2} downwards

R= m(g-a) =70 (10-5)= 70x5= 350N

Reading = 350/ 10 = 35 kg

(c) Acceleration a = 5m s^{-2} upwards

R= m(g+a) = 70(10+5) = 70x 15= 1050N

Reading = 1050/ 10 = 105 kg

(d) when lift falls freely a=g

R= m(g-g) = 0

Reading = 0

Law of Conservation of Momentum

The total momentum of an isolated system of interacting particles is conserved.

Or

When there is no external force acting on a system of particles ,their total momentum remains constant.

Proof of law of conservation of momentum

By Newton's second law of motion , $F=\frac{dp}{dt}$

$$\text{When } F = 0$$

$$\frac{dp}{dt} = 0$$

$$dp = 0\,,$$

$$p=\text{constant}$$

Thus when there is no external force acting on a system of particles, their total momentum remains constant.

Applications of law of conservation of linear momentum

1.Recoil of gun

When a bullet is fired from a gun , the backward movement of gun is called recoil of the gun.

By the law of conservation of momentum,
as the system is isolated,

$$P = \text{constant}$$

Initial momentum = Final momentum

Initial momentum of gun+ bullet system = 0

Final momentum of gun+ bullet system = 0

If p_b and p_g are the momenta of the bullet and gun after firing

$$p_b + p_g = 0$$

$$p_b = -\, p_g$$

The negative sign shows that the gun recoils to conserve momentum.

Expression for Recoil velocity and muzzle velocity

Momentum of bullet after firing , $p_b = mv$

Recoil momentum of the gun after firing , $p_g = MV$

$$p_b = -\, p_g$$

$$mv = -MV$$

Recoil velocity of gun , $V = \frac{-mv}{M}$

Muzzle velocity of bullet , $v = \frac{-MV}{m}$

M= mass of gun, V= recoil velocity of bullet
m= mass of bullet, v=muzzle velocity of bullet

2. The collision of two bodies

Before collision After collision

$$\text{By Newton's second law}, F = \frac{\Delta P}{\Delta t}$$

$$F\Delta t = \Delta P$$

F_{AB} changes the momentum of body A

$$F_{AB}\,\Delta t = p'_A - p_A \text{-----------------(1)}$$

F_{BA} changes the momentum of body B

$$F_{BA}\,\Delta t = p'_B - p_B \text{------------(2)}$$

By Newton's third law

$$F_{AB} = -F_{BA} \qquad \text{-----------(3)}$$

$$p'_A - p_A = -(p'_B - p_B)$$

$$p'_A + p'_B = p_A + p_B$$

Total Final momentum = Total initial momentum
i.e. , the total final momentum of the isolated system equals its total initial momentum.

Equilibrium of a particle

Equilibrium of a particle in mechanics refers to the situation when the net external force on the particle is zero.
According to the first law, this means that, the particle is either at rest or in uniform motion.

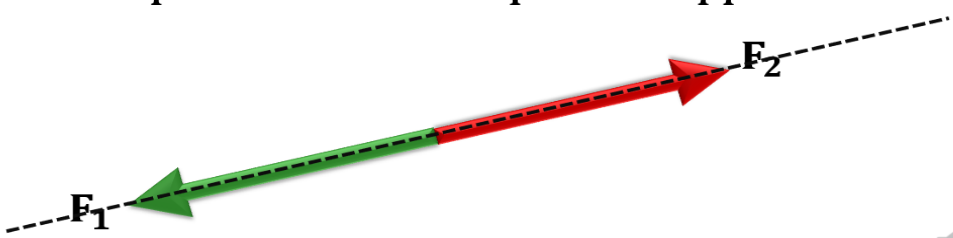
If two forces F_1 and F_2, act on a particle

The equilibrium requires,

$$F_1 + F_2 = 0$$

$$F_1 = -F_2$$

i.e. the two forces on the particle must be equal and opposite.



If three forces F_1, F_2 and F_3, act on a particle

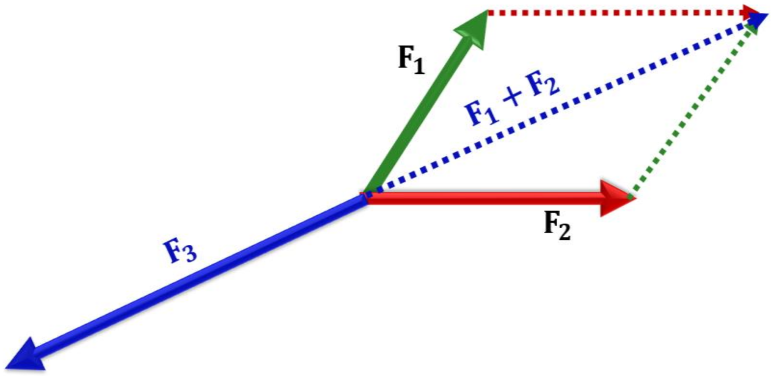
The equilibrium requires,

$$F_1 + F_2 + F_3 = 0$$

$$F_1 + F_2 = -F_3$$

The resultant of any two forces say F_1 and F_2, obtained by the parallelogram law of forces must be equal and opposite to the third force, F_3.

$$F_1 + F_2 = -F_3$$



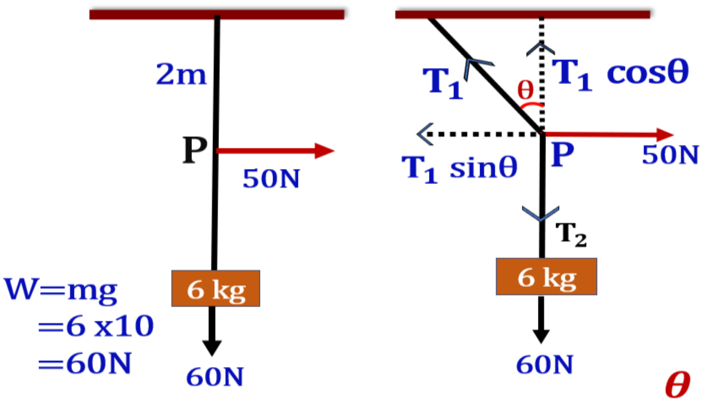
If a number of forces F_1, F_2,........F_n, act on a particle

$$F_1 + F_2 + \dots\dots\dots + F_n = 0$$

Example

A mass of 6 kg is suspended by a rope of length 2 m from the ceiling. A force of 50 N in the horizontal direction is applied at the midpoint P of the rope, as shown. What is the angle the rope makes with the vertical in equilibrium ? (Take $g = 10\ m\ s^{-2}$).

Neglect the mass of the rope.



$$T_1 \cos\theta = 60$$

$$T_1 \sin\theta = 50$$

$$\tan\theta = \frac{50}{60} = \frac{5}{6}$$

$$\theta = \tan^{-1}\left(\frac{5}{6}\right) = 40^0$$

Common Forces in Mechanics

There are two types of forces in mechanics- Contact forces and Non contact forces.

Contact forces

A contact force on an object arises due to contact with some other object: solid or fluid.

Eg: Frictional force, viscous force, air resistance

Non contact forces

A non contact force can act at a distance without the need of any intervening medium.

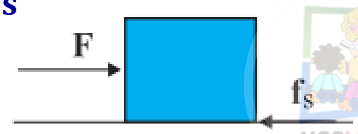
Eg: Gravitational force.

Friction

The force that opposes (impending or actual) relative motion between two surfaces in contact is called frictional force.

There are two types of friction-Static and Kinetic friction

Static friction f_s



- Static friction is the frictional force that acts between two surfaces in contact before the actual relative motion starts. Or Static friction f_s opposes impending relative motion.
  The maximum value of static friction is $(f_s)_{max}$
- The limiting value of static friction $(f_s)_{max}$, is independent of the area of contact.
- The limiting value of static friction $(f_s)_{max}$, varies with the normal force(N)

$$(f_s)_{max} \propto N$$

$$(f_s)_{max} = \mu_s N$$

Where the **constant μ_s is called the coefficient of static friction** and depends only on the nature of the surfaces in contact.

The Law of Static Friction

The law of static friction may thus be written as , $f_s \leq \mu_s N$

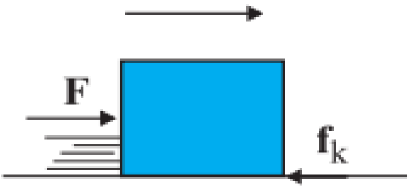
Or

$$(f_s)_{max} = \mu_s N$$

Note:

If the applied force F exceeds $(f_s)_{max}$, the body begins to slide on the surface. When relative motion has started, the frictional force decreases from the static maximum value $(f_s)_{max}$

Kinetic friction f_k



Frictional force that opposes (actual) relative motion between surfaces in contact is called kinetic or sliding friction and is denoted by f_k. .

- Kinetic friction is independent of the area of contact.
- Kinetic friction is nearly independent of the velocity.
- Kinetic friction , f_k varies with the normal force(N)

$$f_k \alpha N$$

$$\mathbf{f_k = \mu_k N}$$

where μ_k the **coefficient of kinetic friction**, depends only on the surfaces in contact.

μ_k is less than μ_s

The Law of Kinetic Friction

The law of kinetic friction can be written as , $\mathbf{f_k = \mu_k N}$

where μ_k the **coefficient of kinetic friction,**

Example

▶ ***Example 5.7*** Determine the maximum acceleration of the train in which a box lying on its floor will remain stationary, given that the co-efficient of static friction between the box and the train's floor is 0.15.

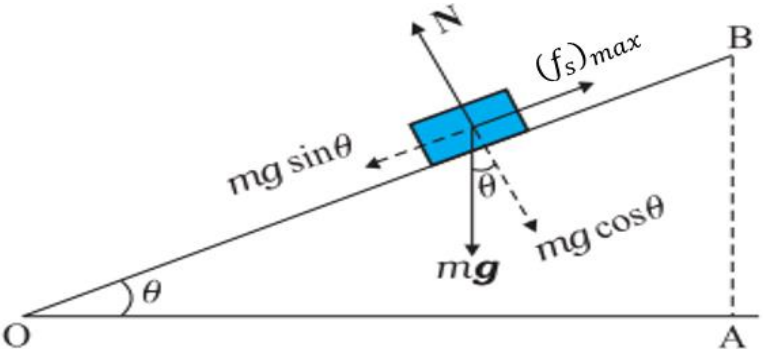
Answer Since the acceleration of the box is due to the static friction,

$$ma = f_s \le \mu_s N = \mu_s\ m g$$

i.e. $a \le \mu_s g$

$$\therefore a_{max} = \mu_s g = 0.15 \times 10 \text{ m s}^{-2}$$
$$= 1.5 \text{ m s}^{-2}$$ ◀

Body on an inclined surface



The forces acting on a block of mass m When it just begins to slide are

(i) the weight mg

(ii) the normal force N

(iii) the maximum static frictional force $(f_s)_{max}$

In equilibrium, the resultant of these forces must be zero.

$$m\,g\sin\theta = (f_s)_{max}$$

$$\text{But } (f_s)_{max} = \mu_s N$$

$$mg\sin\theta = \mu_s N \text{------------}(1)$$

$$m\,g\cos\theta = N\text{------------}(2)$$

$$\text{Eqn}\frac{(1)}{(2)} \text{--------} \frac{mg\sin\theta}{m\,g\cos\theta} = \frac{\mu_s N}{N}$$

$$\boldsymbol{\mu_s = \tan\theta}$$

$$\boldsymbol{\theta = \tan^{-1}\mu_s}$$

This angle whose tangent gives the coefficient of friction is called angle of friction.

> ***Example 5.8*** See Fig. 5.11. A mass of 4 kg rests on a horizontal plane. The plane is gradually inclined until at an angle $\theta = 15°$ with the horizontal, the mass just begins to slide. What is the coefficient of static friction between the block and the surface ?

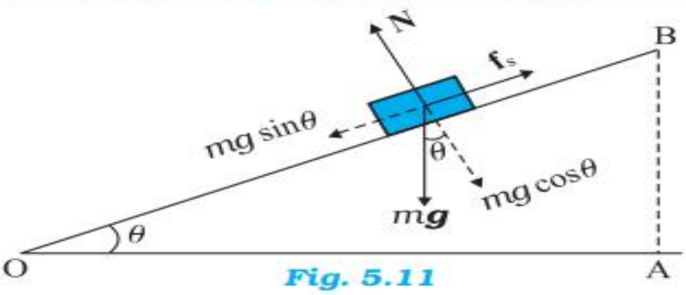


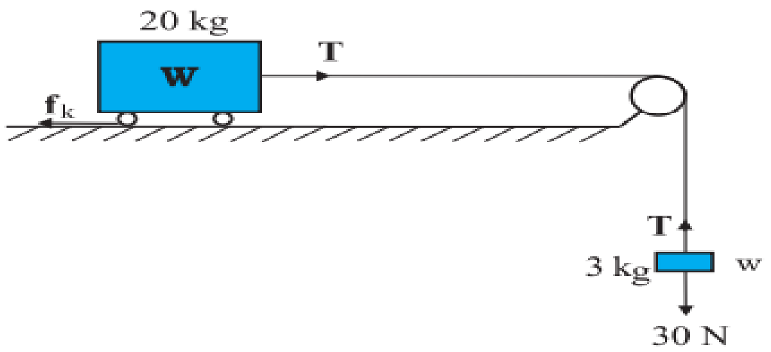
Fig. 5.11

$$\mu_s = \tan\theta$$

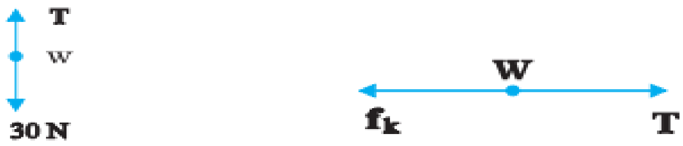
$$\theta = 15°,$$

$$\mu_s = \tan 15° = 0.27$$

Example 5.9 What is the acceleration of the block and trolley system shown in a Fig. 5.12(a), if the coefficient of kinetic friction between the trolley and the surface is 0.04? What is the tension in the string? (Take g = 10 m s⁻²). Neglect the mass of the string.



(a)



Applying second law to motion of the block

$$30 - T = 3a \text{ ---------------(1)}$$

Apply the second law to motion of the trolley

$$T - f_k = 20\,a \text{ ------------(2)}$$

Now $f_k = \mu_k N$,

Here $\mu_k = 0.04$, $N = 20 \times 10 = 200$ N.

Substituting in eq(2)

$$T - 0.04 \times 200 = 20\,a$$

$$T - 8 = 20a \text{-------------(3)}$$

Solving eqns (1) and (3)

$$a = \frac{22}{23}\,\text{m s}^{-2} = 0.96\ \text{m s}^{-2}$$

$$T = 27.1\ \text{N}$$

Rolling Friction

It is the frictional force that acts between the surfaces in contact when one body rolls over the other.

Rolling friction is much smaller than static or sliding friction

Disadvantages of friction

friction is undesirable in many situations, like in a machine with different moving parts, friction opposes relative motion and thereby dissipates power in the form of heat, etc.

Advantages of friction

Friction is a necessary evil .In many practical situations friction is critically needed. Kinetic friction is made use of by brakes in machines and automobiles. We are able to walk because of static friction. It is impossible for a car to move on a very slippery road. On an ordinary road, the friction between the tyres and the road provides the necessary external force to accelerate the car.

Methods to reduce friction

(1)Lubricants are a way of reducing kinetic friction in a machine.
(2)Another way is to use ball bearings between two moving parts of a machine.
(3) A thin cushion of air maintained between solid surfaces in relative motion is another effective way of reducing friction.

Circular Motion

The acceleration of a body moving in a circular path is directed towards the centre and is called centripetal acceleration.

$$a = \frac{v^2}{R}$$

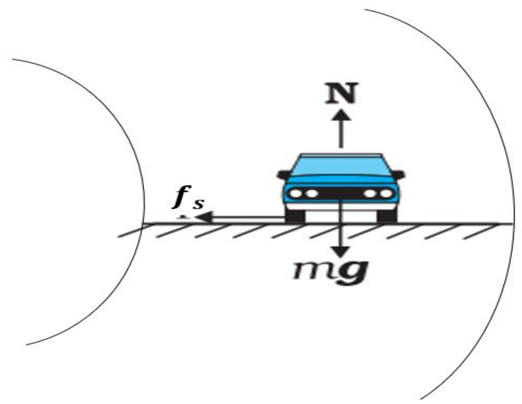
The force f providing centripetal acceleration is called the centripetal force and is directed towards the centre of the circle.

$$f_s = \frac{mv^2}{R}$$

where m is the mass of the body, R is the radius of circle.
For a stone rotated in a circle by a string, the centripetal force is provided by the tension in the string.

The centripetal force for motion of a planet around the sun is the gravitational force on the planet due to the sun.

Motion of a car on a curved level road



Three forces act on the car.

(i) The weight of the car, mg

(ii) Normal reaction, N

(iii) Frictional force, f_s

As there is no acceleration in the vertical direction

$$N = mg$$

The static friction provides the centripetal acceleration

$$f_s = \frac{mv^2}{R}$$

$$\text{But}, f_s \leq \mu_s N$$

$$\frac{mv^2}{R} \leq \mu_s mg \quad (N=mg)$$

$$\mathbf{v^2 \leq \mu_s Rg}$$

$$\mathbf{v_{max} = \sqrt{\mu_s Rg}}$$

This is the maximum safe speed of the car on a circular level road.

Example

A cyclist speeding at 18 km/h on a level road takes a sharp circular turn of radius 3 m without reducing the speed. The co-efficient of static friction between the tyres and the road is 0.1. Will the cyclist slip while taking the turn ?

$$v^2 \leq \mu_s R g$$

$$R = 3 \text{ m}, \quad g = 9.8 \text{ m s}^{-2} \quad , \mu_s = 0.1.$$

$$\mu_s R g = 2.94 \text{ m}^2 \text{ s}^{-2}$$

$$v^2 \leq 2.94 \text{ m}^2 \text{ s}^{-2}$$

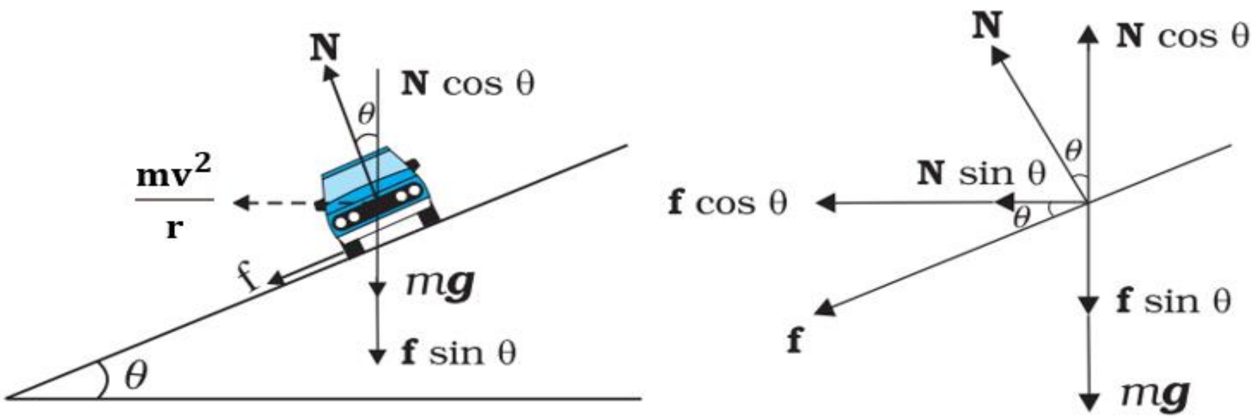
$$\text{Here } v = 18 \text{ km/h} = 5 \text{ m s}^{-1}$$

$$\text{i.e., } v^2 = 25 \text{ m}^2 \text{ s}^{-2}$$

The condition is not obeyed. The cyclist will slip while taking the circular turn.

Motion of a car on a banked road

Raising the outer edge of a curved road above the inner edge is called banking of curved roads.



Since there is no acceleration along the vertical direction, the net force along this direction must be zero.

$$N\cos\theta = mg + f\sin\theta$$

$$N\cos\theta - f\sin\theta = mg \text{ --------------(1)}$$

The centripetal force is provided by the horizontal components of N and f_s.

$$N\sin\theta + f\cos\theta = \frac{mv^2}{R} \text{ --------------(2)}$$

$$\frac{Eqn(1)}{Eqn(2)} \text{ ----- } \frac{N\cos\theta - f\sin\theta}{N\sin\theta + f\cos\theta} = \frac{mg}{\frac{mv^2}{R}}$$

Dividing throughout by $N\cos\theta$

$$\frac{1 - \frac{f}{N}\tan\theta}{\tan\theta + \frac{f}{N}} = \frac{Rg}{v^2}$$

But , $\frac{f}{N} = \mu_s$ for maximum speed

$$\frac{1 - \mu_s\tan\theta}{\tan\theta + \mu_s} = \frac{Rg}{v^2}$$

$$v^2 = \frac{Rg(\mu_s + \tan\theta)}{1 - \mu_s\tan\theta}$$

$$\mathbf{v_{max}} = \sqrt{\frac{Rg(\mu_s + \tan\theta)}{1 - \mu_s\tan\theta}}$$

This is the maximum safe speed of a vehicle on a banked Curved road.

If friction is absent, $\mu_s = 0$

Then Optimum speed, $\mathbf{v_{optimum}} = \sqrt{\mathbf{Rg\tan\theta}}$

Example

A circular racetrack of ra dius 300 m is banked at an angle of 15°. If the coefficient of friction between the wheels of a race-car and the road is 0.2, what is the

(a) optimum speed of the racecar to avoid wear and tear on its tyres, and

(b) maximum permissible speed to avoid slipping ?

r = 300 m ; θ = 15° ; μ_s = 0.2

So the optimum speed becomes,

$$v_o = \sqrt{rg \tan\theta}$$

$$\therefore v_o = \sqrt{300 \times 9.8 \times \tan 15}$$

$$\therefore v_o = \sqrt{300 \times 9.8 \times 0.2679} = 28.1\, m/s$$

Maximum permissible speed on banked road is,

$$v_{max} = \sqrt{\frac{rg(\tan\theta + \mu_s)}{1 - \mu_s \tan\theta}}$$

$$\therefore v_{max} = \sqrt{\frac{300 \times 9.8 \times (\tan 15 + 0.2)}{(1 - 0.2 \times \tan 15)}}$$

$$v_{max} = \sqrt{\frac{300 \times 9.8 \times (\tan 15 + 0.2)}{(1 - 0.2 \times \tan 15)}}$$

$$v_{max} = \sqrt{\frac{300 \times 9.8 \times (0.2679 + 0.2)}{(1 - 0.2 \times 0.2679)}} = 38.1\, m/s$$

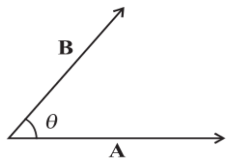
Chapter 5
Work ,Energy and Power

The Scalar Product or Dot Product

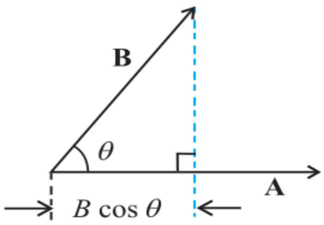
The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}.\vec{B}$
(read A dot B) is defined as

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

where θ is the angle between the two vectors

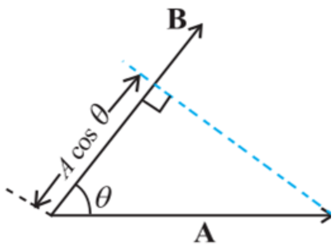


Since A, B and cos θ are scalars, the dot product of A and B is a scalar quantity. Each vector, A and B, has a direction but their scalar product does not have a direction.



$$\vec{A}\cdot\vec{B} = A(B\cos\theta\ )$$

$\vec{A}\cdot\vec{B}$ = magnitude of Ax projection of B onto A



$$\vec{A}\cdot\vec{B} = (A\cos\theta\ )B$$

$\vec{A}\cdot\vec{B}$ = magnitude of Bx projection of Aonto B

Properties of scalar product

- The scalar product follows the commutative law

$$\vec{A}\cdot\vec{B} = \vec{B}.\vec{A}$$

- Scalar product obeys the distributive law

$$\vec{A}\cdot(\vec{B}+\vec{C}\,) = \vec{A}\cdot\vec{B}+\vec{A}.\vec{C}$$

$$\vec{A}.(\lambda\,\vec{B}) = \lambda\,(\vec{A}\cdot\vec{B})$$ where λ is a real number.

- For unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ we have

$$\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}} = 1$$

$$\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\imath}} = 0$$

- For two vectors $\bar{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

$$\bar{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \,.\, (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\mathbf{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

- $\vec{A}\cdot\vec{A} = A_xA_x + A_yA_y + A_z\,A_z$

  $\vec{A}\cdot\vec{A} = {A_x}^2 + {A_y}^2 + {A_z}^2 = A^2$

- $\vec{A}\cdot\vec{A}$ **= A A cos 0 =** A^2

- If $\vec{A}$ and $\vec{B}$ **are perpendicular**

  $\vec{A}\cdot\vec{B} = A\,B\cos 90 = 0$

Example

Find the angle between force $\vec{F} = (3\hat{i} + 4\hat{j} - 5\hat{k})$ unit and displacement $\vec{d} = (5\hat{i} + 4\hat{j} + 3\hat{k})$ unit. Also find the projection of F on d.

$$\vec{F}.\vec{d} = Fd\cos\theta$$

$$\cos\theta = \frac{\vec{F}.\vec{d}}{F\,d} \quad \text{----------(1)}$$

$$\vec{F}.\vec{d} = F_xd_x + F_yd_y + F_zd_z$$

$$= (3x5) + (4x4) + (-5\,x3)$$

$$\mathbf{\vec{F}.\vec{d} = 16\ unit}$$

$$\mathbf{F} = \sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2} = \sqrt{3^2 + 4^2 + (-5)^2}$$

$$= \sqrt{9 + 16 + 25}$$

$$\mathbf{F} = \sqrt{50}\ \mathbf{unit}$$

$$\mathbf{d} = \sqrt{{d_x}^2 + {d_y}^2 + {d_z}^2} = \sqrt{5^2 + 4^2 + 3^2}$$

$$= \sqrt{25 + 16 + 9}$$

$$\mathbf{d} = \sqrt{50}\mathbf{unit}$$

Substituting the values in eq(1)

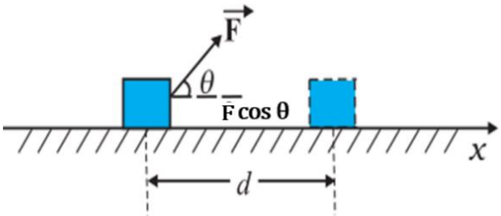
$$\cos\theta = \frac{16}{\sqrt{50}\sqrt{50}} = \frac{16}{50} = 0.32$$

$$\boldsymbol{\theta} = \mathbf{cos}^{-1}\,0.32$$

The projection of F on d = F cosθ=$\sqrt{50}$ x 0.32 **=2.26**

Work

Consider a constant force F acting on an object of mass m. The object undergoes a displacement d in the positive x-direction



The work done by the force is defined to be the product of component of the force in the direction of the displacement and the magnitude of this displacement.

$$W = (F\cos\theta)d$$

$$W = F\,d\cos\theta$$

$$W = \vec{F}\cdot\vec{d}$$

Work can be zero, positive or negative.

Zero Work

The work can be zero,if

(i)the displacement is zero .

When you push hard against a rigid brick wall, the force you exert on the wall does no work.

A weightlifter holding a 150 kg mass steadily on his shoulder for 30 s does no work on the load during this time.

(ii) the force is zero.

A block moving on a smooth horizontal table is not acted upon by a horizontal force (since there is no friction), but may undergo a large displacement.

(iii) the force and displacement are mutually perpendicular

Here $\theta = 90^\circ$, $\cos(90) = 0$.

For the block moving on a smooth horizontal table, the gravitational force mg does no work since it acts at right angles to the displacement.

Positive Work

If θ is between 0° and 90°, cos θ is positive and work positive.

Eg: Workdone by Gravitational force on a freely falling body is positive

Negative work

If θ is between 90° and 180°, cos θ is negative and work negative.

Eg: the frictional force opposes displacement and $\theta = 180^\circ$.

Then the work done by friction is negative ($\cos 180^\circ = -1$).

Units of Work and Energy

- Work and Energy are scalar quantities.
- Work and energy have the same dimensions, $[ML^2T^{-2}]$.
- The SI unit is **kgm^2s^{-2}** or **joule (J)**, named after the famous British physicist James Prescott Joule.

Alternative Units of Work/Energy in J

erg	10^{-7} J
electron volt (eV)	1.6×10^{-19} J
calorie (cal)	4.186 J
kilowatt hour (kWh)	3.6×10^{6} J

Example

Example 6.3 A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposed to the motion. (a) How much work does the road do on the cycle ? (b) How much work does the cycle do on the road ?

Answer Work done on the cycle by the road is the work done by the stopping (frictional) force on the cycle due to the road.

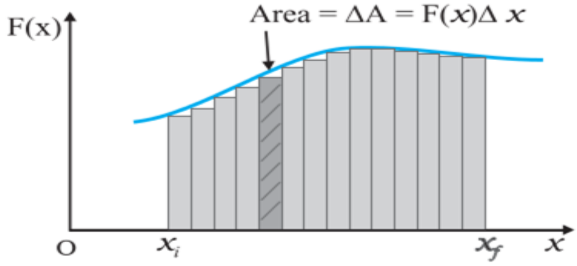
(a) The stopping force and the displacement make an angle of 180° (π rad) with each other. Thus, work done by the road,

$$W_r = Fd\cos\theta$$
$$= 200 \times 10 \times \cos\pi$$
$$= -2000\ \text{J}$$

It is this negative work that brings the cycle to a halt in accordance with WE theorem.

(b) From Newton's Third Law an equal and opposite force acts on the road due to the cycle. Its magnitude is 200 N. However, the road undergoes no displacement. Thus, work done by cycle on the road is zero.

Work done by a Variable Force



If the displacement Δx is small, we can take the force F (x) as approximately constant and the work done is then

$$\Delta W = F(x)\,\Delta x$$

$$W = \int_{x_1}^{x_2} F_{(x)}\, \Delta x$$

In the limit Δx tends to zero

$$\mathbf{W = \int_{x_1}^{x_2} F_{(x)}\, dx}$$

Kinetic Energy

The kinetic energy is the energy possessed by a body by virtue of its motion.

If an object of mass m has velocity v, its kinetic energy K is

$$\mathbf{K = \frac{1}{2} m\bar{v} \cdot \bar{v} = \frac{1}{2} mv^2}$$

Kinetic energy is a scalar quantity.

Example

In a ballistics demonstration a police officer fires a bullet of mass 50.0 g with speed 200 m s-1 on soft plywood of thickness 2.00 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet ?

Answer The initial kinetic energy of the bullet is $mv^2/2 = 1000$ J. It has a final kinetic energy of $0.1 \times 1000 = 100$ J. If v_f is the emergent speed of the bullet,

$$\frac{1}{2} m v_f^2 = 100 \text{ J}$$

$$v_f = \sqrt{\frac{2 \times 100 \text{ J}}{0.05 \text{ kg}}}$$

$$= 63.2 \text{ m s}^{-1}$$

The Work-Energy Theorem

The work-energy theorem can be stated as :**The change in kinetic energy of a particle is equal to the work done on it by the net force.**

Proof

For uniformly accelerated motion

$$v^2 - u^2 = 2as$$

Multiplying both sides by $\frac{1}{2}m$, we have

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mas = Fs$$

$$K_f - K_i = W$$

Change in KE = Work

Potential Energy

Potential energy is the 'stored energy' by virtue of the position or configuration of a body.

- A body at a height h above the surface of earth possesses potential energy due to its position.
- A Stretched or compressed spring possesses potential energy due to its state of strain.

Gravitational potential energy of a body of mass m at a height h above the surface of earth is mgh.

Gravitational Potential Energy , V =mgh

Show that gravitational potential energy of the object at height h, is equal to the kinetic energy of the object on reaching the ground, when the object is released.

PE at a height h, **V = mgh----------(1)**

When the object is released from a height it gains KE

$$K = \frac{1}{2} mv^2$$

$$v^2 = u^2 + 2as$$

$u=0, a=g, s=h$

$$v^2 = 2gh$$

$$K = \frac{1}{2} m \times 2gh$$

K= mgh----------(2)

From eq(1) and (2)

Kinetic energy= Potential energy

Conservative Force

A force is said to be conservative, if it can be derived from a scalar quantity.

$$F = \frac{-dV}{dx} \text{ where V is a scalar}$$

Eg: Gravitational force, Spring force.

- The work done by a conservative force depends only upon initial and final positions of the body
- The work done by a conservative force in a cyclic process is zero

Note: Frictional force , air resistance are non conservative forces.

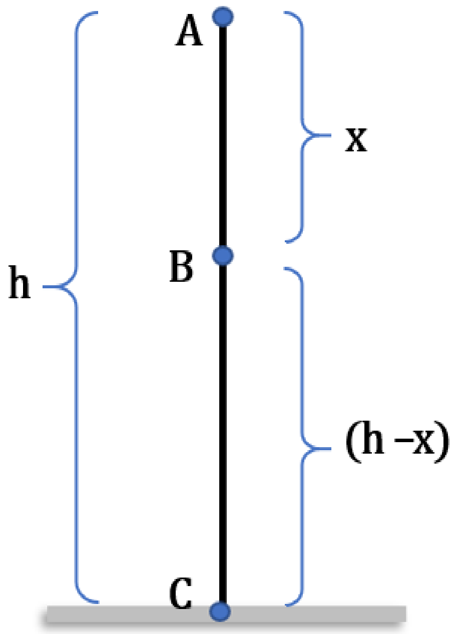
The Conservation of Mechanical Energy

The principle of conservation of total mechanical energy can be stated as,

The total mechanical energy of a system is conserved if the forces, doing work on it, are conservative.

Conservation of Mechanical Energy for a Freely Falling Body

Consider a body of mass m falling freely from a height h



At Point A

PE = mgh

KE = 0 (since v=0)

TE = PE + KE

= mgh + 0

TE = mgh-----------(1)

At Point B

PE = mg (h-x)

KE = ½ mv^2

$v^2 = u^2 + 2as$

u=0, a=g , s= x

$v^2 = 2gx$

KE = ½ m x 2gx

KE= mgx

TE = PE + KE

TE = mg (h-x) + mgx

TE = mgh--------------(2)

At Point C

PE = 0 (Since h=0)

KE = ½ mv^2

$v^2 = u^2 + 2as$

u=0, a=g , s= h

v^2 =2gh

KE = ½ m x 2gh

KE= mgh

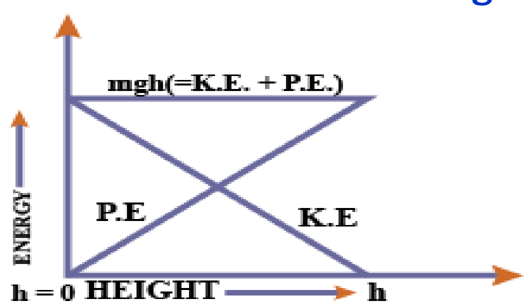
TE = PE + KE

TE = 0 + mgh

TE = mgh--------------(3)

From eqns (1), (2) and (3), it is clear that the total mechanical energy is conserved during the free fall.

Graphical variation of KE and PE with height from ground



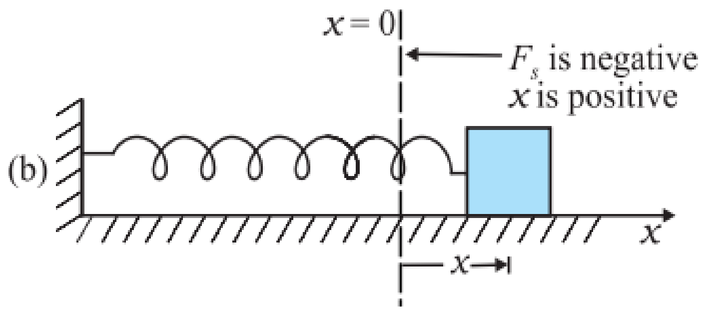
Hooke's Law

Hooke's law states that ,for an ideal spring, the spring force F is proportional displacement x of the block from the equilibrium position.

$$\mathbf{F = -kx}$$

The displacement could be either positive or negative.
The constant k is called the spring constant. Its unit is Nm^{-1}
The spring is said to be stiff if k is large and soft if k is small.

The Potential Energy of a Spring



Consider a block of mass m attached to a spring and resting on a smooth horizontal surface. The other end of the spring is attached to a rigid wall. Let the spring be pulled through a distance x.

Then the spring force $\mathbf{F = -kx}$

The work done by the spring force is

$$W = \int_0^x F\,dx$$

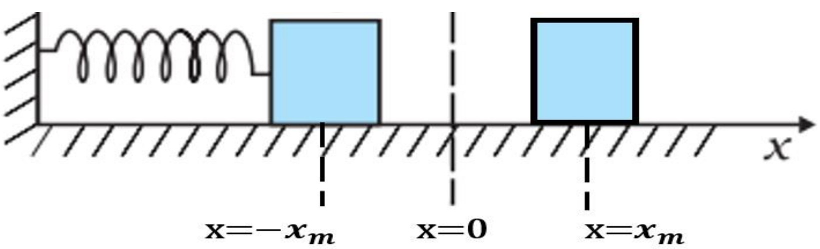
$$W = -\int_0^x kx\,dx$$

$$W = -\frac{1}{2}kx^2$$

This work is stored as potential energy of spring

$$\mathbf{PE = \frac{1}{2}kx^2}$$

Conservation of Mechanical Energy of an Oscillating Spring



Consider a spring oscillating between $-x_m$ and x_m .

At any point x between $-x_m$ *and* x_m , the total mechanical energy of the spring

$$TE = PE + KE$$

$$\frac{1}{2}k\mathbf{x_m}^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

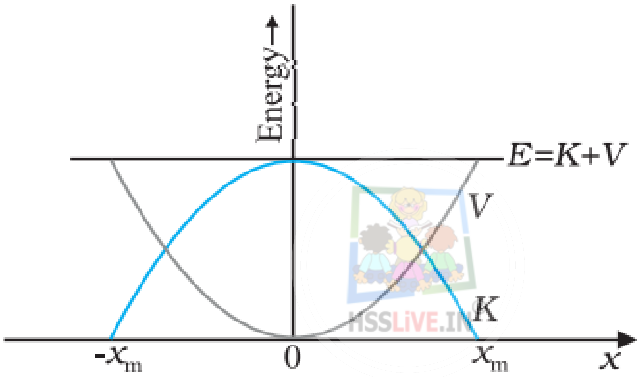
At equilibrium position x=0,

$$\frac{1}{2}k\mathbf{x_m}^2 = \frac{1}{2}mv^2$$

$$\mathbf{v = \sqrt{\frac{k}{m}}x_m}$$

- **At equilibrium position PE is zero and KE is max.**
- **At extreme ends, the PE is maximum and KE is zero.**
- The kinetic energy gets converted to potential energy and vice versa, however, the total mechanical energy remains constant.

Graphical variation of kinetic Energy and potential of a spring



Power

Power is defined as the time rate at which work is done or energy is transferred.

The average power of a force is defined as the ratio of the work, W, to the total time t taken.

$$\mathbf{P_{av} = \frac{W}{t}}$$

The instantaneous power

The instantaneous power is defined as the limiting value of the average power as time interval approaches zero.

$$\mathbf{P = \frac{dW}{dt}}$$

The work done, $dW = F.\,dr.$

$$P = F\,.\frac{dr}{dt}$$

$$\mathbf{P = F\,.\,v}$$

where **v** is the instantaneous velocity when the force is F.

- Power, like work and energy, is a scalar quantity.
- Its dimensions are ML^2T^{-3} .

- SI unit of power is called a watt (W). 1W = 1 J/s
- The unit of power is named after James Watt.
- Another unit of power is the horse-power (hp)

1 hp = 746 W

This unit is still used to describe the output of automobiles, motorbikes, etc

kilowatt hour

Electrical energy is measured in kilowatt hour (kWh).

$$1\text{kWh} = 3.6 \times 10^6 \text{ J}$$

Note:

A 100 watt bulb which is on for 10 hours uses 1 kilowatt hour (kWh) of energy.

Energy = Power x Time

=100 (watt) × 10 (hour)

= 1000 watt hour =

=1 kilowatt hour (kWh)

$= 10^3$ (W) × 3600 (s)

$= 3.6 \times 10^6$ J

Problem

An elevator can carry a maximum load of 1800 kg (elevator + passengers) is moving up with a constant speed of 2 m s–1. The frictional force opposing the motion is 4000 N. Determine the minimum power delivered by the motor to the elevator in watts as well as in horse power.

The downward force on the elevator is F = m g + Frictional Force

= (1800 × 10) + 4000

= 22000 N

Power, P = F. v

= 22000 × 2

= 44000 W

In horse power, power = 44000/746

=59 hp

Collisions

In all collisions the total linear momentum is conserved; the initial momentum of the system is equal to the final momentum of the system. There are two types of collisions Elastic and Inelastic.

Elastic Collisions

The collisions in which both linear momentum and kinetic energy are conserved are called elastic collisions.

Eg: Collision between sub atomic particles

Inelastic Collisions

The collisions in which linear momentum is conserved, but kinetic energy is not conserved are called inelastic collisions. . Part of the initial kinetic energy is transformed into other forms of energy such as heat,sound etc..

Eg: Collision between macroscopic objects

A collision in which the two particles move together after the collision is a perfectly inelastic collision.

Elastic Collisions in One Dimension

If the initial velocities and final velocities of both the bodies are along the same straight line, then it is called a **one-dimensional collision, or head-on collision.**

Before Collision After Collision

Consider two masses m_1 and m_2 making elastic collision in one dimension.

By the conservation of momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \text{ --------------(1)}$$

$$m_1u_1 - m_1v_1 = m_2v_2 - m_2u_2$$

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \text{----------------(2)}$$

By the conservation of kinetic energy

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \text{-----------(3)}$$

$$\frac{1}{2}m_1u_1^2 - \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_2u_2^2$$

$$\frac{1}{2}m_1\,(u_1^2 - v_1^2) = \frac{1}{2}m_2(v_2^2 - u_2^2)$$

$$m_1\,(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad \text{-------------(4)}$$

$$\text{Eqn}\,\frac{(4)}{(2)} \quad \text{------------} \quad \frac{m_1\,(u_1^2 - v_1^2)}{m_1(u_1 - v_1)} = \frac{m_2(v_2^2 - u_2^2)}{m_2(v_2 - u_2)}$$

$$\frac{(u_1+v_1)(u_1-v_1)}{(u_1-v_1)} = \frac{(v_2+u_2)(v_2-u_2)}{(v_2-u_2)}$$

$$u_1 + v_1 = v_2 + u_2 \quad \text{-------------}(5)$$

$$\mathbf{u_1 - u_2 = -(v_1 - v_2)\text{--------}(6)}$$

i.e., relative velocity before collision is numerically equal to relative velocity after collision.

From eqn(5), $v_2 = u_1 + v_1 - u_2$

Substituting in eqn (1)

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2(u_1 + v_1 - u_2)$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2u_1 + m_2v_1 - m_2u_2$$

$$m_1u_1 + m_2u_2 - m_2u_1 + m_2u_2 = m_1v_1 + m_2v_1$$

$$(m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$$

$$\mathbf{v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2} + \frac{2m_2u_2}{m_1+m_2}\text{ -------- }(7)}$$

Similarly, $$\mathbf{v_2 = \frac{(m_2-m_1)u_2}{m_1+m_2} + \frac{2m_1u_1}{m_1+m_2}\text{ ------- }(8)}$$

Case 1 -If two masses are equal, $\mathbf{m_1 = m_2 = \mathit{m}}$

Substituting in eqns (7) and (8)

$$v_1 = \frac{2mu_2}{2m} = u_2$$

$$v_2 = \frac{2mu_1}{2m} = u_1$$

ie.,the bodies will exchange their velocities

Case 2- If one mass dominates, $\mathbf{m_2 >> m_1}$ **and** $\mathbf{u_2} = 0$

$$m_1 + m_2 = m_2 \qquad \mathit{and} \qquad m_1 - m_2 = -m_2$$

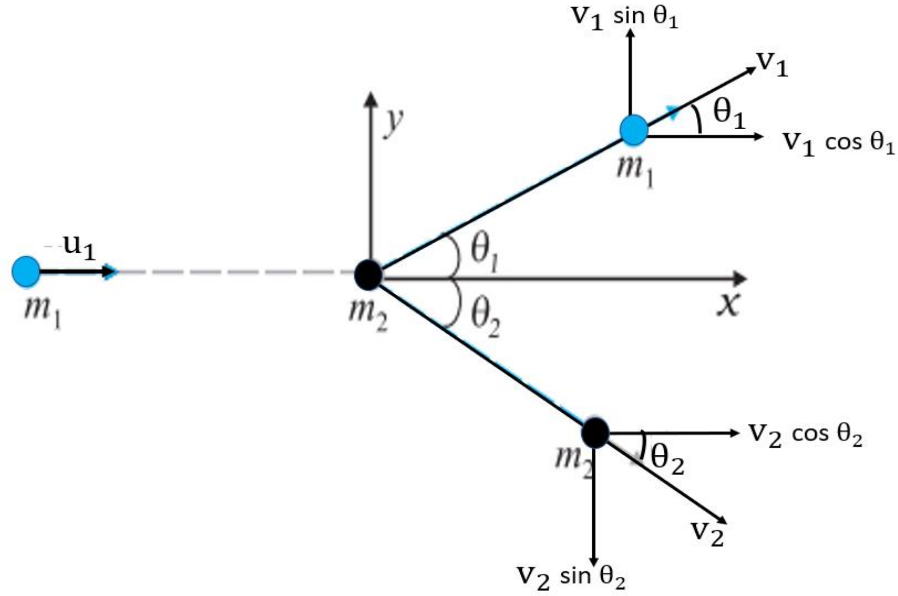
$$v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2} = -\frac{m_2u_1}{m_2} = -u_1$$

$$v_2 = \frac{2m_1u_1}{m_1+m_2} = \frac{2 \times 0 \times u_1}{m_2} = 0$$

(since m_1 is very small , it can be neglected)

The heavier mass comes to rest while the lighter mass reverses its velocity.

Elastic Collisions in Two Dimensions



Consider the elastic collision of a moving mass m_1 with the stationary mass m_2.
Since momentum is a vector ,it has 2 equations in x and y directions.

Equation for conservation of momentum in x direction

$$\mathbf{m_1 u_1 = m_1 v_1 cos\theta_1 + m_2 v_2 cos\theta_2}$$

Equation for conservation of momentum in y direction

$$0 = \mathbf{m_1 v_1 sin\theta_1 - m_2 v_2 sin\,\theta_2}$$

Equation for conservation of kinetic energy,(KE is a scalar quantity)

$$\frac{1}{2}\mathbf{m_1 u_1^2} = \frac{1}{2}\mathbf{m_1 v_1^2} + \frac{1}{2}\mathbf{m_2 v_2^2}$$

Chapter 6
Systems of Particles and Rotational Motion

Rigid Body

Ideally a rigid body is a body with a perfectly definite and unchanging shape. The distances between different pairs of such a body do not change.

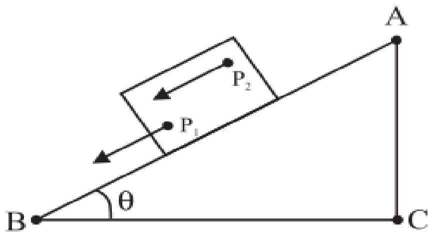
Motion of a rigid body

The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation.
The motion of a rigid body which is pivoted or fixed in some way is rotation.

1)Pure Translational Motion

In pure translational motion at any instant of time every particle of the body has the same velocity.

Eg: A block moving down an inclined plane.

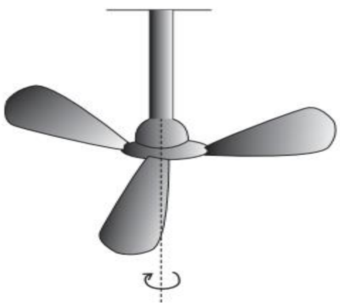


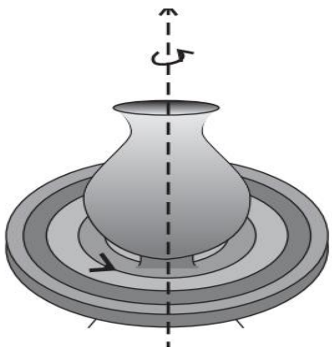
Any point like P1 or P2 of the block moves with the same velocity at any instant of time.

2)Pure Rotational Motion

In pure rotational motion at any instant of time every point in the rotating rigid body has the same angular velocity,but different linear velocity.

i) Rotation about a fixed axis



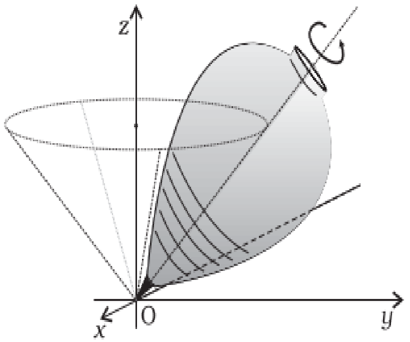


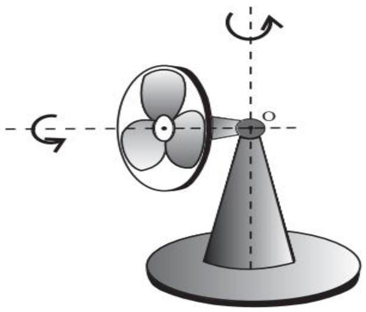
Eg: A ceiling fan A potter's wheel.

The line along which the body is fixed is termed as its axis of rotation.

In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

ii) Rotation about an axis which is not fixed



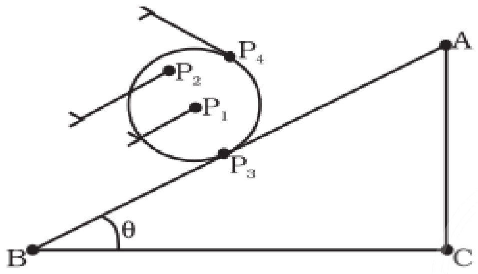


Eg: A spinning top An oscillating table fan

3)Rolling Motion

It is a combination of translational and rotational motion.

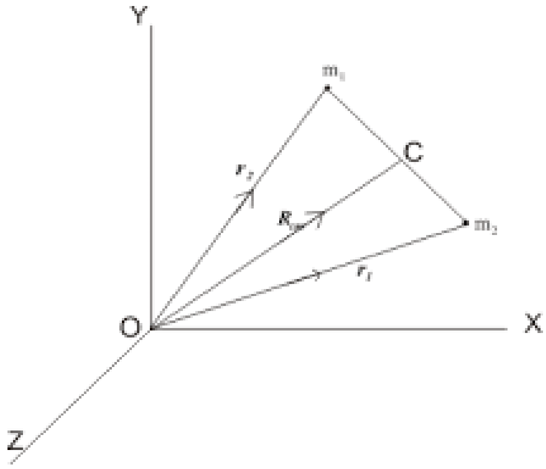
Eg A solid cylinder moving down an inclined plane.



Points P1, P2, P3 and P4 have different velocities at any instant of time. In fact, the velocity of the point of contact P3 is zero at any instant, if the cylinder rolls without slipping.

Centre Of Mass

The centre of is a hypothetical point where the entire mass of an object may be assumed to be concentrated to visualise its motion.



Consider a two particle system. Let C be the centre of mass which is at a distancev X from origin.

$$\vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$$

$$\vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2}{M} \quad \text{where } M=m_1+m_2$$

x coordinate of centre of mass $X = \frac{m_1x_1+m_2x_2}{m_1+m_2}$

y coordinate of centre of mass $Y = \frac{m_1y_1+m_2y_2}{m_1+m_2}$

z coordinate of centre of mass $Z = \frac{m_1z_1+m_2z_2}{m_1+m_2}$

If we have n particles of masses $m_1, m_2, \ldots m_n$

$$\vec{\mathbf{R}} = \frac{\mathbf{m_1\vec{r}_1+m_2\vec{r}_2+\cdots\ldots\ldots+m_n\vec{r}_n}}{\mathbf{M}} \quad \text{--------(1)}$$

$$\vec{\mathbf{R}} = \frac{\mathbf{\sum m_i\vec{r}_i}}{\mathbf{M}} \qquad \text{where } \mathrm{M} = \mathrm{m_1 + m_2 + \ldots\ldots + m_n}$$

If the origin is chosen to be the centre of mass then $\vec{R} = 0$

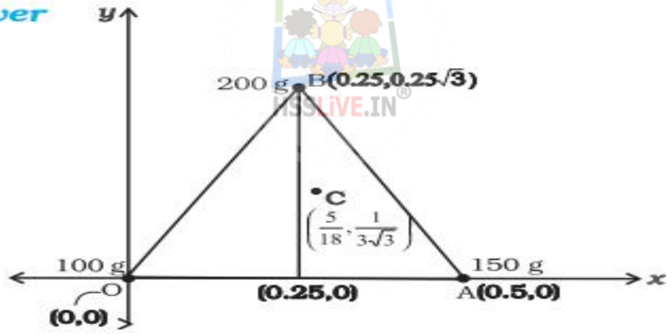
$$0 = \frac{\sum \mathrm{m_i}\vec{\mathrm{r}}_\mathrm{i}}{\mathrm{M}}$$

$$\sum \mathrm{m_i}\vec{\mathrm{r}}_\mathrm{i} = 0$$

Example

Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Answer



$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{[100(0) + 150(0.5) + 200(0.25)] \text{ g m}}{(100 + 150 + 200) \text{ g}}$$

$$= \frac{75 + 50}{450} \text{m} = \frac{125}{450} \text{m} = \frac{5}{18} \text{m}$$

$$Y = \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}$$

$$Y = \frac{[100(0) + 150(0) + 200(0.25\sqrt{3})] \text{ g m}}{450 \text{ g}}$$

$$= \frac{50\sqrt{3}}{450} \text{m} = \frac{\sqrt{3}}{9} \text{m} = \frac{1}{3\sqrt{3}} \text{m}$$

Motion of Centre of Mass

Position vector of centre of mass

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{M}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots \ldots\ldots + m_n\vec{r}_n}{M} \quad \text{--------(1)}$$

where $M = m_1 + m_2 + \ldots\ldots + m_n$

Velocity of centre of mass

Differentiating

$$\frac{d}{dt}\vec{R} = \frac{d}{dt}\left\{\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots \ldots\ldots + m_n\vec{r}_n}{M}\right\}$$

$$\vec{V} = \frac{m_1 \frac{d}{dt}\vec{r}_1 \quad + m_2 \frac{d}{dt}\vec{r}_2 + \cdots \ldots .. + m_n \frac{d}{dt}\vec{r}_n}{M}$$

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots \ldots\ldots + m_n\vec{v}_n}{M} \quad \text{-------------(2)}$$

Acceleration of centre of mass

Differentiating

$$\frac{d}{dt}\vec{V} = \frac{d}{dt}\left\{\frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots \ldots\ldots + m_n\vec{v}_n}{M}\right\}$$

$$\vec{A} = \frac{m_1 \frac{d}{dt}\vec{v}_1 \quad + m_2 \frac{d}{dt}\vec{v}_2 + \cdots \ldots .. + m_n \frac{d}{dt}\vec{v}_n}{M}$$

$$\vec{A} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots\ldots + m_n\vec{a}_n}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Force on centre of mass

Acceleration of centre of mass

$$\vec{A} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots \ldots .. + m_n\vec{a}_n}{M}$$

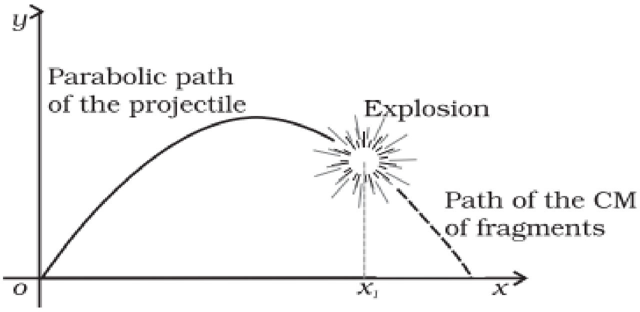
$$M\vec{A} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots \ldots .. + m_n\vec{a}_n$$

$$\vec{\mathbf{F}}_{\mathbf{ext}} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots \ldots .. + m_n\vec{a}_n$$

$$\vec{\mathbf{F}}_{\mathbf{ext}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \ldots \ldots .. + \vec{\mathbf{F}}_n$$

$$\vec{\mathbf{F}}_{\mathbf{ext}} = M\vec{A}$$

The centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.



The centre of mass of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion.

Linear Momentum of centre of mass

Velocity of centre of mass

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots \ldots\ldots + m_n\vec{v}_n}{M}$$

$$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots \ldots \ldots .. + m_n\vec{v}_n$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \cdots \ldots \ldots .. + \vec{p}_n$$

Law of Conservation of Momentum for a System of Particles

If Newton's second law is extended to a system of particles,

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

When the sum of external forces acting on a system of particles is zero

$$\vec{F}_{ext} = 0$$

$$\frac{d\vec{P}}{dt} = 0$$

$$\vec{P} = \text{constant}$$

Thus, when the total external force acting on a system of particles is zero, the total linear momentum of the system is constant. This is the law of conservation of the total linear momentum of a system of particles.

$$\text{But } \vec{P} = M\vec{V}$$

$$M\vec{V} = \text{constant}$$

$$\vec{V} = constant$$

When the total external force on the system is zero the velocity of the centre of mass remains constant or the CM of the system is in uniform motion.

Vector Product or Cross product of Two Vectors

Vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$$

where A and B are magnitudes of $\vec{A}$ and $\vec{B}$

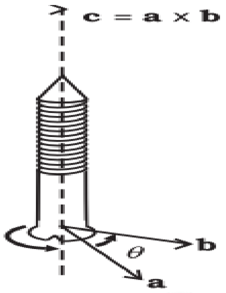
θ is the angle between $\vec{A}$ and $\vec{B}$

$\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$

The direction of $\vec{A} \times \vec{B}$ is given by right hand screw rule or right hand rule

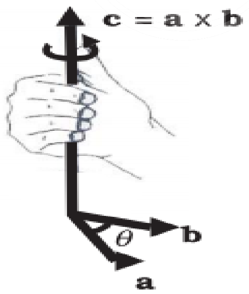
Right hand screw rule

If we turn the head of screw in the direction from $\vec{A}$ to $\vec{B}$,then the tip of the screw advances in the direction of $\vec{A} \times \vec{B}$



Right hand rule

if the fingers of right hand are curled up in the direction from $\vec{A}$ to $\vec{B}$, then the stretched thumb points in the direction of $\vec{A} \times \vec{B}$



- The vector product is not commutative

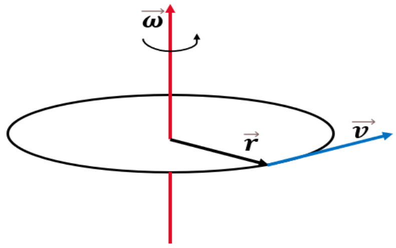
$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

- Vector product obeys distributive law

$$\vec{A} \times (\overrightarrow{B + C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

- $\vec{A} \times \vec{A} = \vec{0}$
- $\hat{\imath} \times \hat{\imath} = 0$, $\hat{\jmath} \times \hat{\jmath} = 0$, $\hat{k} \times \hat{k} = 0$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$
- $\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$, $\hat{\imath} \times \hat{k} = -\hat{\jmath}$

Angular Velocity and its Relation with Linear Velocity



The angular velocity is a vector quantity. $\vec{\omega}$ is directed along the fixed axis as shown.

The linear velocity of the particle is

$$\vec{v} = \vec{\omega} \times \vec{r}$$

It is perpendicular to both $\vec{\omega}$ and $\vec{r}$ and is directed along the tangent to the circle described by the particle.

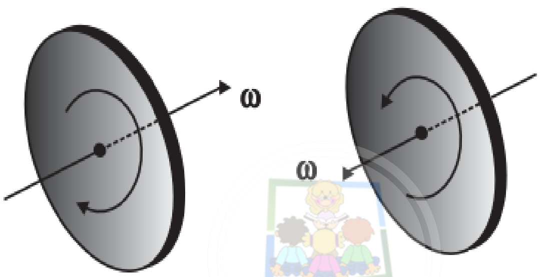


Figure shows the direction of angular velocity when the body rotates in clockwise and anti clockwise direction.

For rotation about a fixed axis, the direction of the vector ω does not change with time. Its magnitude may change from instant to instant. For the more general rotation, both the magnitude and the direction of ω may change from instant to instant.

Angular acceleration

Angular acceleration $\vec{\alpha}$ is defined as the time rate of change of angular velocity.

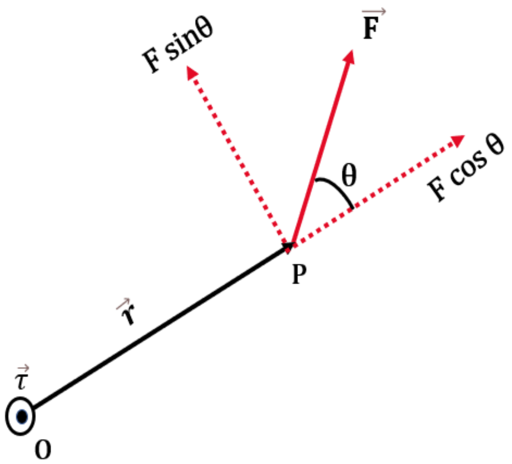
$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

If the axis of rotation is fixed, the direction of ω and hence, that of α is fixed. In this case the vector equation reduces to a scalar equation

$$\alpha = \frac{d\omega}{dt}$$

Torque or Moment of Force

The rotational analogue of force is torque or moment of force .



If a force $\vec{F}$ acts on a single particle at a point P whose position with respect to the origin O is $\vec{r}$,then torque about origin o is

$$\vec{\tau} = r\,F\,\sin\theta$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Torque has dimensions $M\,L^2T^{-2}$
- Its dimensions are the same as those of work or energy.
- It is a very different physical quantity than work.
- Moment of a force is a vector, while work is a scalar.
- The SI unit of moment of force is Newton-metre (Nm)

The magnitude of the moment of force may be written

$$\tau = (r\sin\theta)\,F = r_{\perp}\,F$$

$$\tau = r\,(F\sin\theta) = r\,F_{\perp}$$

where $r_{\perp} = r\sin\theta$ is the perpendicular distance of the line of action of F form the origin and $F_{\perp} = F\sin\theta$ is the component of F in the direction perpendicular to r.

Angular momentum of a particle

Angular momentum is the rotational analogue of linear momentum.

Angular momentum is a vector quantity. It could also be referred to as moment of (linear) momentum.

$$\vec{l} = \vec{r} \times \vec{p}$$

$$\vec{l} = rp\sin\theta$$

Relation connecting Torque and Angular momentum

$$\vec{l} = \vec{r} \times \vec{p}$$

Differentiating

$$\frac{d\vec{l}}{dt} = \frac{\mathrm{d}}{\mathrm{dt}}(\vec{\mathrm{r}} \times \vec{\mathrm{p}})$$

$$\frac{d\vec{l}}{dt} = \frac{\mathrm{d}\vec{\mathrm{r}}}{\mathrm{dt}} \times \vec{\mathrm{p}} + \vec{\mathrm{r}} \times \frac{\mathrm{d}\vec{\mathrm{p}}}{\mathrm{dt}}$$

$$\vec{\mathrm{p}} = \mathrm{m}\vec{\mathrm{v}} \quad , \frac{\mathrm{d}\vec{\mathrm{r}}}{\mathrm{dt}} = \vec{\mathrm{v}} \, , \frac{\mathrm{d}\vec{\mathrm{p}}}{\mathrm{dt}} = \vec{\mathrm{F}}$$

$$\frac{d\vec{l}}{dt} = \vec{\mathrm{v}} \times \mathrm{m}\vec{\mathrm{v}} + \vec{\mathrm{r}} \times \vec{\mathrm{F}}$$

$$\vec{\mathrm{v}} \times \vec{\mathrm{v}} = 0 \, , \ (\vec{\mathrm{r}} \times \vec{\mathrm{F}} = \vec{\tau})$$

$$\frac{d\vec{l}}{dt} = 0 + \vec{\tau}$$

$$\boldsymbol{\vec{\tau} = \frac{d\vec{l}}{dt}}$$

Thus, the time rate of change of the angular momentum of a particle is equal to the torque acting on it.

This is the rotational analogue of the equation $\vec{\mathrm{F}} = \frac{\mathrm{d}\vec{\mathrm{p}}}{\mathrm{dt}}$, which expresses Newton's second law for the translational motion of a single particle.

Relation connecting Torque and Angular momentum for a system of particles

$$\boldsymbol{\vec{\tau} = \frac{d\vec{L}}{dt}}$$

$$\text{where } \vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_n$$

Law of Conservation of Angular momentum

For a system of particles

$$\vec{\tau}_{\mathrm{ext}} = \frac{\mathrm{d}\vec{\mathrm{L}}}{\mathrm{dt}}$$

If external torque $\vec{\tau}_{\mathrm{ext}} = 0$,

$$\frac{\mathrm{d}\vec{\mathrm{L}}}{\mathrm{dt}} = 0$$

$$\boldsymbol{\vec{\mathrm{L}} = \text{constant}}$$

If the total external torque on a system of particles is zero, then the total angular momentum of the system is conserved i.e, remains constant.

Example

Find the torque of a force $7\hat{i} + 3\hat{j} - 5\hat{k}$ about the origin. The force acts on a particle whose position vector is $\hat{i} - \hat{j} + \hat{k}$.

$$\vec{\tau} = \vec{r} \text{ x } \vec{F}$$

$$\vec{\tau} = (\hat{i} - \hat{j} + \hat{k}) \text{ x} (7\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\begin{array}{cc} & \begin{array}{ccc} + & - & + \end{array} \\ \vec{\tau} = & \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{vmatrix} \end{array}$$

$$\vec{\tau} = \hat{i}\,[(-1 \text{ x } -5) - (3 \text{ x } 1)] \;\; - \hat{j}\,[(1\text{x}{-5}) - (7\text{x}1)] \;\; + \hat{k}\,[(1\text{x}3) - (7\text{x}{-1})]$$

$$\vec{\tau} = \hat{i}\,[5 - 3] \;\; - \hat{j}\,[-5 - 7] \;\; + \hat{k}\,[3 - -7]$$

$$\vec{\tau} = 2\hat{i} + 12\,\hat{j} + 10\hat{k}$$

Equilibrium of a Rigid Body

A rigid body is said to be in mechanical equilibrium, if it is in both translational equilibrium and rotational equilibrium.
i.e, for a body in mechanical equilibrium its linear momentum and angular momentum are not changing with time.

Translational Equilibrium

When the total external force on the rigid body is zero, then the total linear momentum of the body does not change with time and the body will be in translational equilibrium .

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{0}$$

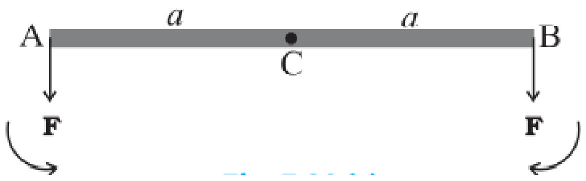
Rotational Equilibrium

When the total external torque on the rigid body is zero, the total angular momentum of the body does not change with time and the body will be in rotational equilibrium .

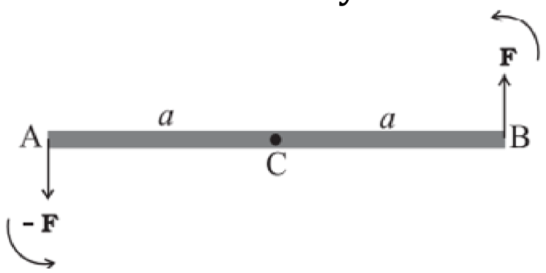
$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \ldots + \boldsymbol{\tau}_n = \sum_{i=1}^{n} \boldsymbol{\tau}_i = \mathbf{0}$$

Partial equilibrium

A body may be in partial equilibrium, i.e., it may be in translational equilibrium and not in rotational equilibrium, or it may be in rotational equilibrium and not in translational equilibrium.



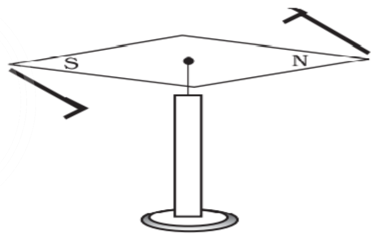
Here net torque is zero and the body is in rotational equilibrium.
Net force is not zero and the body is not in traslational equilibrium



Here net torque is not zero and the body will not be rotational equilibrium.
Net force is zero and the body will be in traslational equilibrium.

Couple

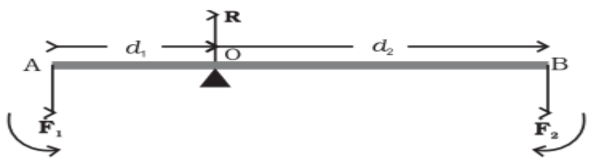
A pair of equal and opposite forces with different lines of action is known as a couple. A couple produces rotation without translation.



Our fingers apply a couple to turn the lid

The Earth's magnetic field exerts equal and opposite forces on the poles of a compass needle. These two forces form a couple.

Principles of Moments



The lever is a system in mechanical equilibrium.
For rotational equilibrium the sum of moments must be zero,

$$\mathbf{d_1F_1 - d_2F_2 = 0}$$

The equation for the principle of moments for a lever is

$$\mathbf{d_1F_1 = d_2F_2}$$

load arm × load = effort arm × effort

$$\text{Mechanical Advantage MA} = \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

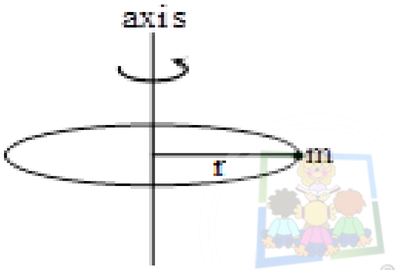
Centre of gravity

The Centre of gravity of a body is the point where the total gravitational torque on the body is zero.

- The centre of gravity of the body coincides with the centre of mass. For a body is small, g does not vary from one point of the body to the other. Then the centre of gravity of the body coincides with the centre of mass.
- If the body is so extended that g varies from part to part of the body, then the centre of gravity and centre of mass will not coincide.

Moment of Inertia

Moment of Inertia is the rotational analogue of mass.
Moment of inertia is a measure of rotational inertia



The moment of inertia of a particle of mass m rotating about an axis is

$$I = mr^2$$

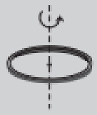
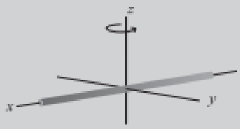
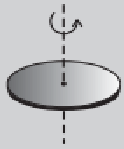
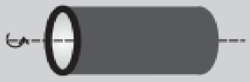
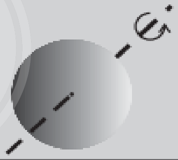
The moment of inertia of a rigid body is

$$I = \sum_{i=1}^{n} m_i r_i^2$$

The moment of inertia of a rigid body depends on

- The mass of the body, its shape and size
- Distribution of mass about the axis of rotation
- The position and orientation of the axis of rotation.

Moments of Inertia of some regular shaped bodies about specific axes

Z	Body	Axis	Figure	I
(1)	Thin circular ring, radius R	Perpendicular to plane, at centre		MR^2
(2)	Thin circular ring, radius R	Diameter		$MR^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$ML^2/12$
(4)	Circular disc, radius R	Perpendicular to disc at centre		$MR^2/2$
(5)	Circular disc, radius R	Diameter		$MR^2/4$
(6)	Hollow cylinder, radius R	Axis of cylinder		MR^2
(7)	Solid cylinder, radius R	Axis of cylinder		$MR^2/2$
(8)	Solid sphere, radius R	Diameter		$2MR^2/5$

Rotational Kinetic energy

Cosider a particle of mass m rotating about an axis of radius r with angular velocity ω

The kinetic energy of motion of this particle is

$$kE = \frac{1}{2}mv^2$$

But $v = r\,\omega$

$$kE = \frac{1}{2}mr^2\omega^2$$

$I = mr^2$

$$\textbf{Rotational kE} = \frac{1}{2}\mathbf{I}\boldsymbol{\omega}^2$$

Radius of Gyration (k)

The radius of gyration can be defined as the distance of a mass point from the axis of roatation whose mass is equal to the whole mass of the body and whose moment of inertia is equal to moment of inertia of the whole body about the axis.

If K is the radius of gyration, we can write

$$I = Mk^2$$

$$k = \sqrt{\frac{I}{M}}$$

Flywheel

The machines, such as steam engine and the automobile engine, etc., that produce rotational motion have a disc with a large moment of inertia, called a flywheel. Because of its large moment of inertia, the flywheel resists the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for the passengers on the vehicle.

Kinematics of Rotational Motion about a Fixed Axis

The kinematical equations of linear motion with uniform (i.e. constant) acceleration

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2ax$$

The corresponding kinematic equations for rotational motion with uniform angular acceleration are:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

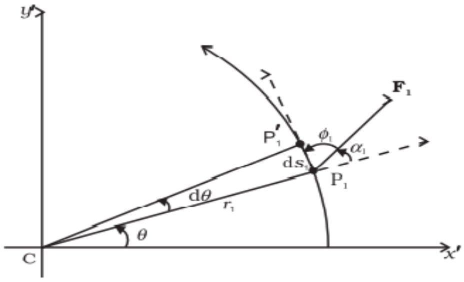
and $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Dynamics of Rotational Motion about a Fixed Axis

Comparison of Translational and Rotational Motion

	Linear Motion	Rotational Motion about a Fixed Axis
1	Displacement x	Angular displacement θ
2	Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
3	Acceleration $a = dv/dt$	Angular acceleration $\alpha = d\omega/dt$
4	Mass M	Moment of inertia I
5	Force $F = Ma$	Torque $\tau = I\alpha$
6	Work $dW = F\,ds$	Work $W = \tau\, d\theta$
7	Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
8	Power $P = F v$	Power $P = \tau\omega$
9	Linear momentum $p = Mv$	Angular momentum $L = I\omega$

Work done by a torque



Work done by a force F acting on a particle of a body rotating about a fixed axis

$$dW = F.ds$$
$$dW = Fds\cos\varphi$$
but $\varphi + \alpha = 90$, $\varphi = 90 - \alpha$
$\cos(90 - \alpha) = \sin\alpha$
$$dW = F(r\,d\theta)\sin\alpha$$
$$dW = r\,F\sin\alpha\,d\theta$$
$$dW = \tau\,d\theta$$
$$\mathbf{W = \tau\,\theta}$$

Instantaneous power by a Torque

$$P = \frac{dW}{dt}$$
But, $dW = \tau\,d\theta$
$$P = \tau\frac{d\theta}{dt}$$
$$\mathbf{P = \tau\omega}$$

Angular Momentum in Case of Rotation about a Fixed Axis.(or) Relation Connecting Angular Momentum And Moment Of Inertia

Angular momentum of a particle, $\vec{l} = \vec{r} \times \vec{p}$

For a system of particles , $\vec{L} = \sum \vec{l}$

$$\vec{L} = \sum \vec{r} \times \vec{p}$$
$$\vec{L} = \sum rp\sin 90\,\hat{k}$$
$$\vec{L} = \sum rp\,\hat{k}$$
$$\vec{L} = \sum rmv\,\hat{k} \qquad (p=mv)$$
$$\vec{L} = \sum rm(r\omega)\,\hat{k} \qquad (v=r\omega)$$
$$\vec{L} = \sum mr^2\omega\,\hat{k}$$
$$\vec{L} = I\omega\hat{k}$$
$$\mathbf{\vec{L} = I\vec{\omega}}$$

Relation Connecting Torque and Angular Acceleration

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

But $\vec{L} = I\vec{\omega}$

$$\vec{\tau} = \frac{dI\vec{\omega}}{dt}$$

$$\vec{\tau} = I\,\frac{d\vec{\omega}}{dt}$$

$$\boldsymbol{\vec{\tau} = I\,\vec{\alpha}}$$

Conservation of angular momentum

If the external torque is zero, angular momentum is constant.

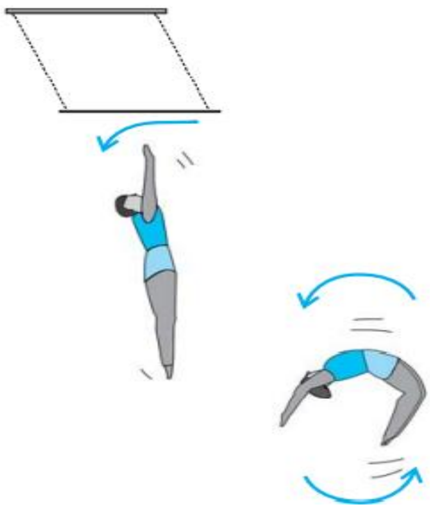
$$L = \text{constant}$$

$$\mathbf{I\omega = constant}$$

When I increases ,ω decreases and vice versa, so that Iω is constant.

While the chair is rotating with considerable angular speed , if you stretch your arms horizontally, moment of inertia(I) increases and as a result, the angular speed(ω) is reduced.

If you bring back your arms closer to your body, moment of inertia(I) decreases and as a result, the angular speed(ω) increases again.



A circus acrobat and a diver take advantage of this principle.

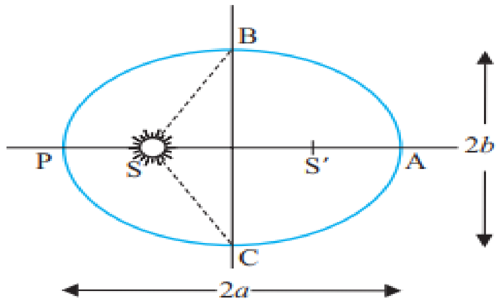
Also, skaters and classical, Indian or western, dancers performing a pirouette on the toes of one foot display 'mastery' over this principle.

Chapter 7
Gravitation

Kepler's Laws

1.Law of orbits

All planets move in elliptical orbits with the Sun situated at one of the foci of the ellipse.



PA is the major axis
BC is the minor axis

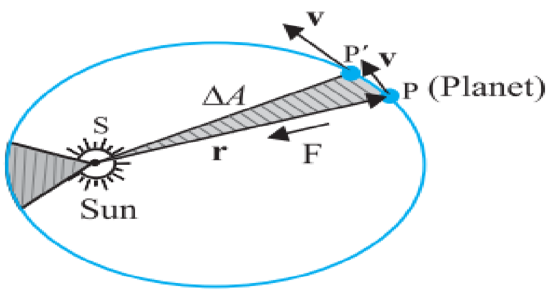
2.Law of areas

The line that joins any planet to the sun sweeps equal areas in equal intervals of time. i.e, areal velocity $\frac{\Delta\vec{A}}{\Delta t}$ is constant

The planets move slower when they are farther from the sun than when they are nearer.

The law of areas is a consequence of conservation of angular momentum.

Proof



The area swept out by the planet of mass m in time interval Δt is

$$\Delta\vec{A} = \frac{1}{2}\left(\vec{r} \times \vec{v}\Delta t\right)$$

$$\vec{p} = m\vec{v},$$

$$\vec{v} = \frac{\vec{p}}{m}$$

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\left(\vec{r} \times \frac{\vec{p}}{m}\right)$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{L}}{2m}$$

For a central force, which is directed along r, angular momentum, $\vec{L}$ is a constant.

$$\frac{\Delta\vec{A}}{\Delta t} = \text{constant}$$

This is the law of areas.

3.Law of periods

The square of the time period of revolution of a planet is proportional to the cube of the semi- major axis of the ellipse traced out by the planet.

$$T^2 \propto a^3$$

Universal Law of Gravitation

Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them .

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant.

In vector Form

$$\vec{F} = G\frac{m_1 m_2}{r^2}(-\hat{r}) = -G\frac{m_1 m_2}{r^2}\hat{r}$$

$\hat{r}$ is the unit vector from m_1 to m_2.

The gravitational force is attractive, as the force $\vec{F}$ is along – r.

By Newton's third law the, gravitational force $\vec{F}_{12}$ on the body 1 due to 2 and $\vec{F}_{21}$ on the body 2 due to 1 are related as $\vec{F}_{12} = -\vec{F}_{21}$.

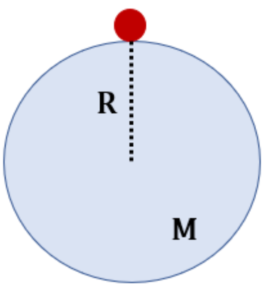
The Gravitational Constant

The value of the gravitational constant G was determined experimentally by English scientist Henry Cavendish in 1798.

$$G = 6.67\times10^{-11}\ N\ m^2/kg^2$$

Acceleration due to gravity of the Earth

Consider a body of mass m on the surface of earth of mass M and radius R. The gravitational force between body and earth is given by



$$F = \frac{GMm}{R^2} \quad \text{-----------(1)}$$

By Newton's second law

$$F=mg \quad \text{-----------(2)}$$

where g is acceleration due to gravity

From Eq (1) and (2)

$$mg = \frac{GMm}{R^2}$$

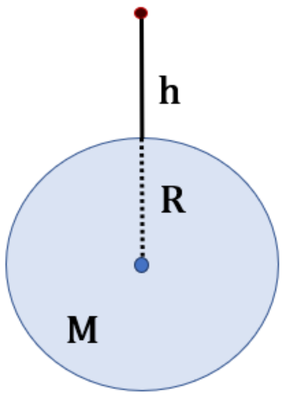
$$g = \frac{GM}{R^2}$$

Acceleration due to gravity is independent of mass of the body.

The average value of g on the surface of earth is $9.8ms^{-2}$.

Acceleration due to gravity below and above the surface of earth

1.Acceleration due to gravity at a height h above the surface of the earth.



Acceleration due to gravity on the surface of earth

$$g = \frac{GM}{R^2} \text{------------}(1)$$

Acceleration due to gravity at a height above the surface of earth

$$g_h = \frac{GM}{(R+h)^2} \quad \text{----------}(2)$$

for , h << R, $g_h = \frac{GM}{R^2(1+\frac{h}{R})^2}$

$$g_h = \frac{GM}{R^2}(1 + \frac{h}{R})^{-2}$$

substituting from eq(1) , $g_h = g(1 + \frac{h}{R})^{-2}$

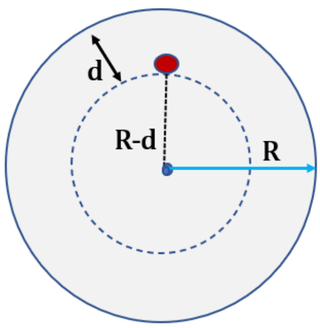
Using binomial expression and neglecting higher order terms.

$$\mathbf{g_h \cong g\,(1 - \frac{2h}{R})}$$

Thus, as we go above earth's surface, the acceleration due gravity decreases by a factor $(\mathbf{1} - \frac{\mathbf{2h}}{\mathbf{R}})$

2.Acceleration due to gravity at a depth d below the surface of the earth

We assume that the entire earth is of uniform density. Then mass of earth



Mass =volume x density

$$M = \frac{4}{3}\pi R^3 \rho \text{------------}(1)$$

Acceleration due to gravity on the surface of earth

$$g = \frac{GM}{R^2} \quad \text{-------------}(2)$$

Substituting the value of M in eq(2)

$$g = \frac{G}{R^2}(\frac{4}{3}\pi R^3 \rho)$$

$$g = \frac{4}{3}\pi R \rho G \text{---------------}(3)$$

Acceleration due to gravity at a depth d below the surface of earth

$$g_d = \frac{4}{3}\pi(R-d)\rho G \text{---------}(4)$$

$$\frac{eq(4)}{eq(3)} \text{------} \quad \frac{g_d}{g} = \frac{\frac{4}{3}\pi(R-d)\rho G}{\frac{4}{3}\pi R \rho G}$$

$$\frac{g_d}{g} = \frac{(R-d)}{R}$$

$$\mathbf{g_d = g(1 - \frac{d}{R})}$$

Thus, as we go down below earth's surface, the acceleration due gravity decreases by a factor $(\mathbf{1-\frac{d}{R}})$

- The value of acceleration due to earth's gravity is maximum on its surface and decreases whether you go up or down.
- At the centre of earth acceleration due to earth's gravity is **zero.**

Example

At what height the value of acceleration due to gravity will be half of that on surface of earth. (Given the radius of earth R =6400km)

.

$$g_h = g(1+\frac{h}{R})^{-2}$$

$$g_h = \frac{g}{2}$$

$$\frac{g}{2} = g(1+\frac{h}{R})^{-2}$$

$$\frac{1}{2} = (1+\frac{h}{R})^{-2}$$

$$2 = (1+\frac{h}{R})^{2}$$

$$\sqrt{2} = 1+\frac{h}{R}$$

$$\frac{h}{R} = \sqrt{2}\text{-}1$$

$$h = (\sqrt{2}\text{-}1)\ R$$

$$h = (1.414\ \text{-}1)\ 6400 = 2650\ \text{km}$$

Example

Calculate the value of acceleration due to gravity at a height equal to half of the radius of earth.

$$g_h = \frac{GM}{(R+h)^2}$$

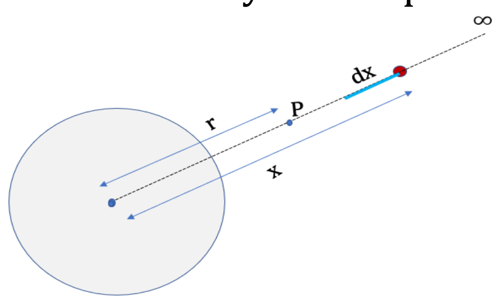
$$h = \frac{R}{2}$$

$$g_h = \frac{GM}{\left(R+\frac{R}{2}\right)^2} = \frac{GM}{\left(\frac{3}{2}R\right)^2}$$

$$= \frac{GM}{\frac{9}{4}R^2} = \frac{4}{9}\frac{GM}{R^2} = \frac{4}{9}\ g$$

Gravitational Potential Energy

Gravitational potential energy at point is defined as the work done in displacing the particle from infinity to that point without acceleration.



Gravitational force on a mass m at a distance x

$$F = \frac{GMm}{x^2}$$

The work done to give a displacement dx to the mass

$$dW = Fdx$$

$$dW = \frac{GMm}{x^2}dx$$

Total work done to move the mass from ∞ **to r**

$$W = \int_{\infty}^{r} \frac{GMm}{x^2} dx$$

$$W = GMm \int_{\infty}^{r} \frac{1}{x^2} dx$$

$$W = GMm \left[\frac{-1}{x}\right]_{\infty}^{r}$$

$$W = -GMm \left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$W = \frac{-GMm}{r}$$

This work is is stored as gravitational PE in the body.

$$\mathbf{U = \frac{-GMm}{r}}$$

Gravitational Potential

The gravitational potential due to the gravitational force of the earth is defined as the potential energy of a particle of unit mass at that point.
The gravitational Potential energy of a bodyof mass m at a distance r

$$U = \frac{-GMm}{r}$$

For unit mass m=1

So gravitational potential, $\mathbf{V = \frac{-GM}{r}}$

Escape speed

The minimum speed required for an object to reach infinity i.e. to escape from the earth's gravitational pull is called escape speed.
Let the body thrown from the surface of earth to infinity.

Total initial energy of the body

$$TE = KE + PE$$

$$TE = \frac{1}{2}mv_i^2 - \frac{GMm}{R} \text{-------(1)}$$

Total final energy, $$TE = \frac{1}{2}mv_f^2 + 0 \text{------------(2)}$$

By conservation of energy TE is constant.

$$\frac{1}{2}mv_i^2 - \frac{GMm}{R} = \frac{1}{2}mv_f^2$$

RHS is always always a positive quantity with minimum value zero
Taking the minimum value

$$\frac{1}{2}mv_i^2 - \frac{GMm}{r} = 0$$

$$\frac{1}{2}mv_i^2 = \frac{GMm}{R}$$

$$v_i^2 = \frac{2GM}{R}$$

$$\mathbf{v_e = \sqrt{\frac{2GM}{R}}}$$

But $g = \frac{GM}{R^2}$, $GM = gR^2$

$$v_e = \sqrt{\frac{2gR^2}{R}}$$

$$\mathbf{v_e = \sqrt{2gR}}$$

Escape velocity is independent of mass of the body.
Escape speed (or escape velocity) on the surface of earth is 11.2km/s

Moon has no atmosphere. Why?

The escape speed of moon is about 2.3 km/s. which is less than the average speed of gas molecules of moon. Thus gas molecules escape from surface of moon and it has no atmosphere.

Earth Satellites

Earth satellites are objects which revolve around the earth.
Their motion is very similar to the motion of planets around the Sun and hence Kepler's laws of planetary motion are equally applicable to them.

Satellites are of two types (1) Natural satellites and artificial satellites. Moon is the natural satellite of earth whose time period of revolution is 27.3 days.
Artificial satellites are used for telecommunication, geophysics and meteorology etc.

Orbital Speed

The speed with which a satellites revolves around earth is called orbital speed.

Consider a satellite of mass m in a circular orbit of a distance (R + h) from the centre of the earth. **The necessary centripetal force for revolution is provided by gravitational force between earth and satellite.**

$$F_{gravitational} = \frac{GMm}{(R+h)^2}$$

$$F_{centripetal} = \frac{mv^2}{R+h}$$

$$F_{centripetal} = F_{gravitational}$$

$$\frac{mv^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$v^2 = \frac{GM}{(R+h)}$$

$$v_o = \sqrt{\frac{GM}{(R+h)}}$$

Thus orbital velocity v_o decreases as height ,h increases.
If the satellite is very close to earth (R+h) ≈R

$$v_o = \sqrt{\frac{GM}{R}}$$

But $g = \frac{GM}{R^2}$, $GM = gR^2$

$$v_o = \sqrt{gR}$$

Relation Connecting Escape Velocity and Orbital Velocity

Orbital Velocity , $v_0 = \sqrt{\frac{GM}{R}}$ or $v_o = \sqrt{gR}$

Escape Velocity , $v_e = \sqrt{\frac{2GM}{R}}$ or $v_e = \sqrt{2gR}$

$$v_e = \sqrt{2}\, v_o$$

Escape Velocity = $\sqrt{2}$ x Orbital Velocity

Period of a Satellite

Period of a satellite is the time required for a satellite to complete one revolution around the earth in a fixed orbit.

$$\text{Time}=\frac{\text{Distance}}{speed}$$

For one revolution

$$\text{Period T}=\frac{\text{circumference of the orbit}}{orbital\ speed}$$

$$\text{T}=\frac{2\pi\,(R+h)}{\sqrt{\frac{\text{GM}}{(\text{R+h})}}}$$

$$\mathbf{T=2\pi\sqrt{\frac{(R+h)^3}{GM}}}$$

If the satellite is very close to earth (R+h) ≈R

$$\mathbf{T=2\pi\sqrt{\frac{R^3}{GM}}}$$

But $g=\frac{GM}{R^2}$, $GM = gR^2$

$$\text{T}=2\pi\sqrt{\frac{R^3}{gR^2}}$$

$$\mathbf{T=2\pi\sqrt{\frac{R}{g}}}$$

If we substitute the numerical values, $g = 9.8\ \text{m}s^{-2}$ and $R = 6400$ km.,

$$\text{T}=2\pi\sqrt{\frac{6400\ x\ 10^3}{9.8}}$$

$$T_o=85 \text{ minutes}$$

Proof of Kepler's third law

Period of a satellite, $\text{T}= 2\pi\sqrt{\frac{(\text{R+h})^3}{\text{GM}}}$

$$\text{T}^2= 4\pi^2\,\frac{(\text{R+h})^3}{\text{GM}}$$

$$\text{T}^2= \frac{4\pi^2}{\text{GM}}(\text{R} + \text{h})^3$$

$$\text{T}^2=\text{constant x }(\text{R} + \text{h})^3$$

$$\text{T}^2 \propto (\text{R} + \text{h})^3$$

$$\mathbf{T^2 \propto a^3}$$

Which is Kepler's Law of Periods.

Weighing the Earth :

You are given following data: g = 9.81s^{-2} m, $R_E = 6.37\times10^6$ m, the distance to the moon R = 3.84× 10^8 m and the time period of the moon's revolution is 27.3 days. Obtain mass of the Earth M_E in two different ways.

First method

$$g=\frac{GM}{R^2}$$

$$M_E = \frac{g\,R_E^2}{G} = \frac{9.81\times(6.37\times10^6)^2}{6.67\times10^{-11}}$$

$$=5.97\times\ 10^{24}\,\text{kg.}$$

Second method

$$T=2\pi\sqrt{\frac{R^3}{GM}}$$

$$T^2 = \frac{4\pi^2R^3}{G\,M_E}$$

$$M_E = \frac{4\pi^2R^3}{G\,T^2} = \frac{4\times3.14\times3.14\times(3.84)^3\times10^{24}}{6.67\times10^{-11}\times(27.3\times24\times60\times60)^2}$$

$$= 6.02\times10^{24}\,\text{kg}$$

Both methods yield almost the same answer

Energy of an orbiting satellite

$$\text{KE}=\frac{1}{2}\text{mv}_o^2$$

$$v_o = \sqrt{\frac{GM}{R+h}}$$

$$v_o^2 = \frac{GM}{R+h}$$

$$\text{KE}=\frac{1}{2}\text{m}\,x\,\frac{GM}{R+h}$$

$$\text{KE}=\frac{GMm}{2(R+h)}$$

$$\text{PE} = \frac{-GMm}{R+h}$$

$$\text{Energy} = \text{KE} + \text{PE}$$

$$E = \frac{GMm}{2(R+h)} + \frac{-GMm}{R+h}$$

$$\mathbf{E = \frac{-GMm}{2(R+h)}}$$

The total energy of an circularly orbiting satellite is negative, which means that the satellite is bound to the planet .If the total energy is positive or zero, the object escapes to infinity. Satellites are always at finite distance from the earth and hence their energies cannot be positive or zero.

Chapter 8
Mechanical Properties of Solids

A solid has definite shape and size. In order to change (or deform) the shape or size of a body, a force is required.

Elasticity

The property of a body, by virtue of which it tends to regain its original size and shape when the applied force is removed, is known as elasticity and such substances are called elastic .

Eg: Steel, Rubber

Steel is more elastic than rubber.

Plasticity

Some substances have no tendency to regain their previous shape on the removal of deforming force and they get permanently deformed. Such substances are called plastic and this property is called plasticity.

Eg:Putty and mud

Stress and Strain

When a force is applied on body, it is deformed to a small or large extent depending upon the nature of the material of the body and the magnitude of the deforming force. When a body is subjected to a deforming force, a restoring force is developed in the body. This restoring force is equal in magnitude but opposite in direction to the applied force.

Stress

The restoring force per unit area is known as stress.

If F is the force applied and A is the area of cross section of the body,

$$\text{Stress} = \frac{F}{A}$$

The SI unit of stress is N m^{-2} or pascal (Pa)

Dimensional formula of stress is [$ML^{-1}T^{-2}$]

Strain

Strain is defined as the **fractional change in dimension.**

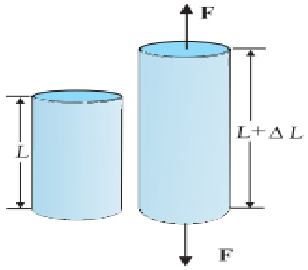
$$\text{Strain} = \frac{\text{Change in dimension}}{\text{Original dimension}}$$

Strain has no unit and dimension.

There are three ways in which a solid may change its dimensions when an external force acts on it. As a result there are three types of stress and strain.

1. Longitudinal Stress and Longitudinal Strain
2. Shearing Stress and Shearing Strain
3. Hydraulic Stress and Hydraulic Strain (Volume Strain)

1.Longitudinal Stress and Longitudinal Strain



Longitudinal stress is defined as the restoring force per unit area when force is applied normal to the cross-sectional area of a cylinder.

$$\text{Longitudinal stress} = \frac{F}{A}$$

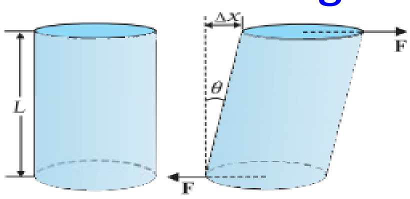
If the cylinder is stretched the stress is called **tensile stress** and If the cylinder is compressed it is called **compressive stress**.

Longitudinal strain is defined as the ratio of change in length(ΔL) to original length(L) of the body .

$$\text{Longitudinal strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$\text{Longitudinal strain} = \frac{\Delta L}{L}$$

2.Shearing Stress and Shearing Strain



Shearing stress is defined as the restoring force per unit area when a tangential force is applied on the cylinder.

$$\text{Shearing stress} = \frac{F}{A}$$

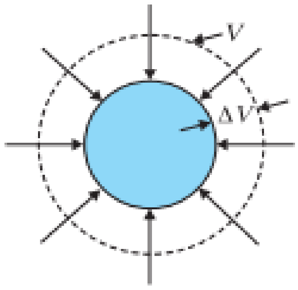
Shearing strain is defined as the ratio of relative displacement of the faces Δx to the length of the cylinder L

$$\text{Shearing strain} = \frac{\Delta x}{L} = \tan\theta$$

Usually θ is very small, tan θ ≈ θ

$$\text{Shearing strain} = \theta$$

3.Hydraulic Stress and Hydraulic strain (Volume Strain)



When a solid sphere placed in the fluid, the force applied by the fluid acts in perpendicular direction at each point of the surface.

The restoring force per unit area of solid sphere, placed in the fluid is called hydraulic stress.

$$\text{Hydraulic stress} = \frac{F}{A} = -P \text{ (pressure)}$$

The negative sign indicates that when pressure increases, the volume decreases.

Volume strain(hydraulic strain) is defined as the ratio of change in volume (ΔV) to the original volume (V).

$$\text{Volume strain} = \frac{\text{Change in volume}}{\text{Original volume}}$$

$$\text{Volume strain} = \frac{\Delta V}{V}$$

Hooke's Law

For small deformations the stress is directly proportional to strain. This is known as Hooke's law.

$$\text{Stress} \propto \text{Strain}$$

$$\text{stress} = k \times \text{strain}$$

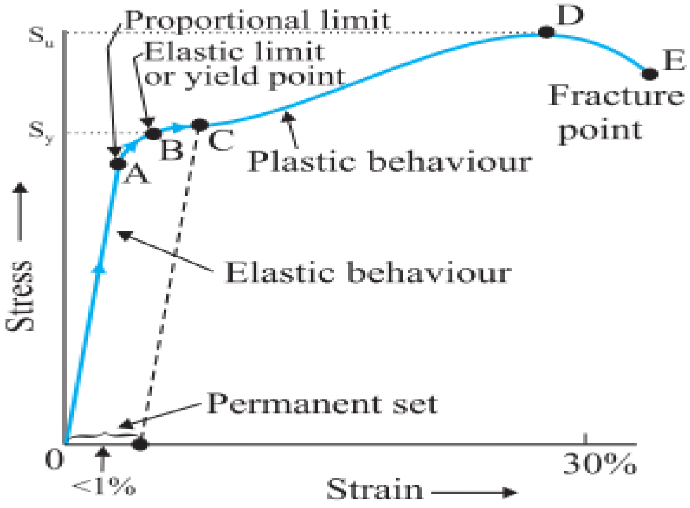
$$\frac{\text{Stress}}{\text{strain}} = k$$

where k is a constant and is known as **Modulus of Elasticity.**

- The SI unit of modulus of elasticity is N m^{-2} or pascal (Pa)
  (same as that of stress,since strain is unitless)
- Dimensional formula is [$ML^{-1}T^{-2}$]

Stress-Strain Curve

A typical stress-strain curve for a metal is as shown in figure:



In the region from O to A

The curve is linear. In this region, stress is proportional to strain i.e, Hooke's law is obeyed.

In the region from A to B

Stress and strain are not proportional, i.e, Hooke's law is not obeyed. Nevertheless, the body is still elastic.

The point B in the curve is known as yield point or elastic limit.

The stress corresponding to yield point is known as yield strength (S_y) of the material.

In the region from B to D

Beyond the point B ,the strain increases rapidly even for a small change in the stress. When the load is removed, at some point C between B and D, the body does not regain its original dimension. The material is said to have a permanent set. The material shows plastic behaviour in this region.
The point D on the graph is the ultimate tensile strength (S_u) of the material.

In the region from D to E

Beyond this point D, additional strain is produced even by a reduced applied force and fracture occurs at point E.
The point E is called **Fracture Point.**
If the ultimate strength and fracture points **D and E are close,** the material is said to be **brittle.**
If **D and E are far apart,** the material is said to be **ductile.**

Elastomers

Substances like tissue of aorta, rubber etc. which can be stretched to cause large strains are called elastomers.

- Eventhough elastic region is very large, the material does not obey Hoke's law for most of the regions.
- There is no well defined plastic region.

Elastic Moduli

The ratio of stress and strain, called modulus of elasticity. Depending upon the types of stress and strain there are three moduli of elasticity.

1. **Young's Modulus(Y)**
2. **Shear Modulus or Rigidity Modulus (G)**
3. **Bulk modulus(B)**

1.Young's Modulus(Y)

The ratio of longitudinal stress to longitudinal strain is defined as Young's modulus of the material .

$$Y=\frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$Y=\frac{\frac{F}{A}}{\frac{\Delta L}{L}}$$

$$\mathbf{Y=\frac{FL}{A\,\Delta L}}$$

If $F=mg$ and $A=\pi r^2$

$$\mathbf{Y=\frac{mgL}{\pi r^2\,\Delta L}}$$

- SI unit of Young's modulus is N m^{-2} or Pa.
- For metals Young's moduli are large.
- Steel is more elastic than rubber as the Young's modulus of steel is large.
- Wood, bone, concrete and glass have rather small Young's moduli.

Why steel is preferred in heavy-duty machines and in structural designs?

Young's modulus of steel is greater than that of copper, brass and aluminium. It means that steel is more elastic than copper, brass and aluminium. It is for this reason that steel is preferred in heavy-duty machines and in structural designs.

Example

A structural steel rod has a radius of 10 mm and a length of 1.0 m. A 100 kN force stretches it along its length. Calculate (a) stress, (b) elongation, and (c) strain on the rod. Young's modulus, of structural steel is 2.0×10^{11} N m^{-2}

(a) $$\text{stress} = \frac{F}{A} = \frac{F}{\pi r^2}$$

$$= \frac{100\ x\ 10^3}{3.14\ x\ (10\ x\ 10^{-3})^2} = \frac{100\ x\ 10^3}{3.14\ x\ 10^{-4}}$$

$$= 3.18\ \text{x}10^8\ \text{N}m^{-2}$$

(b) $$Y = \frac{FL}{A\,\Delta L}$$

$$\Delta L = \frac{(\frac{F}{A})L}{Y} = \frac{3.18\ \text{x}10^8\ x\ 1}{2 \times 10^{11}}$$

$$= 1.59\ \text{x}10^{-3}\ \text{m}$$

$$= 1.59\ \text{mm}$$

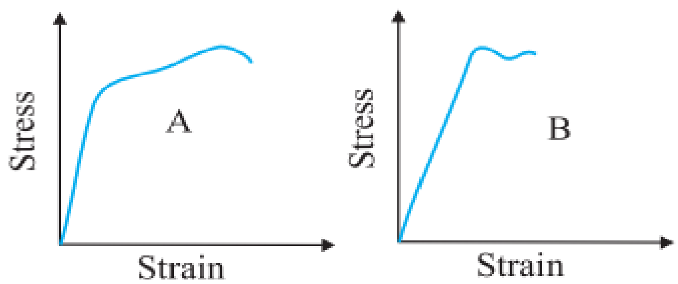
(c) $$\text{Strain} = \frac{\Delta L}{L}$$

$$= \frac{1.59\ \text{x}10^{-3}}{1}$$

$$= 1.59\ \text{x}10^{-3}\ \text{m}$$

Example

The stress-strain graphs for materials A and B are shown in Figure.



The graphs are drawn to the same scale.

(a) Which of the materials has the greater Young's modulus?
(b) Which of the two is the stronger material?
(c) Which of the two materials is more ductile?

(a) Young's modulus $Y = \frac{\text{stress}}{\text{strain}}$ = slpoe of the graph.
Slope of graph for material A is greater than that of B.
So materials A has the greater Young's modulus.

(b) Strength of a material is determined by the amount of stress required to cause fracture.
The fracture point is greater for material A.
So material A is stronger than B

(c) The fracture point is far apart for material A than B.
So material A is more ductile than B.

2.Shear Modulus or Rigidity Modulus(G)

The ratio of shearing stress to the corresponding shearing strain is called the shear modulus or Rigidity modulus of the material .

$$G = \frac{\text{Shearing stress}}{\text{Shearing strain}}$$

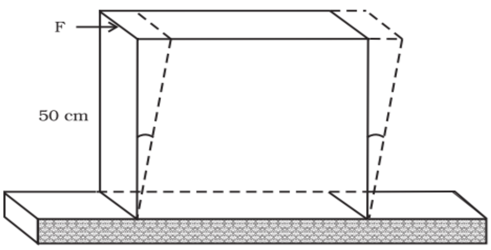
$$G = \frac{\frac{F}{A}}{\frac{\Delta x}{L}} = \frac{\frac{F}{A}}{\theta}$$

$$G = \frac{\boldsymbol{F}}{\mathbf{A}\,\boldsymbol{\theta}}$$

SI unit of shear modulus is N m^{-2} or Pa.
Shear modulus is generally less than Young's modulus.
For most materials $G \approx Y/3$

Example

A square lead slab of side 50 cm and thickness 10 cm is subject to a shearing force (on its narrow face) of 9.0×10^4 N. The lower edge is riveted to the floor. How much will the upper edge be displaced? Given shear modulus,
$G = 5.6 \times 10^9 \text{Nm}^{-2}$



$$\text{Stress} = \frac{F}{A} = \frac{9.0 \times 10^4}{0.5 \text{ x} 0.1} = \frac{9.0 \times 10^4}{0.05}$$
$$= 1.80 \times 10^6 \text{ N } m^{-2}$$
$$G = \frac{\text{stress}}{\frac{\Delta x}{L}}$$
$$\Delta x = \frac{stress\ x\ L}{G}$$
$$= \frac{1.80 \times 10^6\ x\ 0.5}{5.6 \times 10^9} = 1.6 \times 10^{-4} \text{ m} = 0.16 \text{ mm}$$

3.Bulk Modulus(B)

The ratio of hydraulic stress to the corresponding hydraulic strain is called bulk modulus.

$$B = \frac{\text{Hydraulic stress}}{\text{Hydraulic strain}}$$

$$B = \frac{\frac{F}{A}}{\frac{\Delta V}{V}} = \frac{-P}{\frac{\Delta V}{V}}$$

$$\mathbf{B = \frac{-PV}{\Delta V}}$$

SI unit of Bulk modulus is N m^{-2} or Pa.

The negative sign indicates that when pressure increases, the volume decreases. That is, if p is positive, ΔV is negative. Thus for a system in equilibrium, the value of bulk modulus B is always positive.

Compressibility(k)

The reciprocal of the bulk modulus is called compressibility.

$$k = \frac{1}{B}$$
$$\mathbf{k = \frac{-1}{P} \frac{\Delta V}{V}}$$

- The bulk moduli for solids are much larger than for liquids, which are again much larger than the bulk modulus for gases (air).
- Thus solids are least compressible whereas gases are most compressible.

Example

The average depth of Indian Ocean is about 3000 m. Calculate the fractional compression, $\Delta V/V$, of water at the bottom of the ocean, given that the bulk modulus of water is 2.2×10^9 N m^{-2}. (Take $g = 10$ m s^{-2})

$$G = \frac{-P}{\frac{\Delta V}{V}}$$

$$\frac{\Delta V}{V} = \frac{P}{G}$$

$$P = h\rho g = 3000 \text{ x}1000 \text{ x}10 = 3\text{x}10^7 \text{ N m}^{-2}$$

$$\frac{\Delta V}{V} = \frac{3\text{x}10^7}{2.2 \times 10^9} = 1.36 \times 10^{-2}$$

Poisson's ratio

When a material is stretched in one direction, it tends to compress in the direction perpendicular to that of force application and vice versa.
For example, a rubber band tends to become thinner when stretched.

The strain in the direction of applied force is called longitudinal strain.

$$\text{Longitudinal strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$= \frac{\Delta L}{L}$$

The strain perpendicular to the direction of applied force is called lateral strain.

$$\text{Lateral Strain} = \frac{\text{Change in diameter}}{\text{Original diameter}}$$

$$= \frac{\Delta d}{d}$$

The ratio of lateral strain to longitudinal strain is called Poisson's ratio.

$$\text{Poisson's Ratio } \sigma = \frac{\text{Lateral Strain}}{\text{Longitudnal Strain}}$$

$$\sigma = \frac{\frac{\Delta d}{d}}{\frac{\Delta L}{L}}$$

$$\boldsymbol{\sigma = \frac{\Delta d}{\Delta L} \text{ x} \frac{L}{d}}$$

Poisson's ratio has no unit and dimension.

Elastic Potential Energy in a Stretched String

The workdone to deform a body against the inter atomic force is stored as elastic PE.

For a string, work done for a small elongation dl

$$dW = F.d\,l$$

$$W = \int_0^l F.\,dl$$

But $Y = \frac{FL}{A\,l}$,

l is the eleogation of string

$$F = \frac{YA\,l}{L}$$

$$W = \int_0^l \frac{YA\,l}{L}.\,dl$$

$$= \frac{YA}{L}\ x\ \frac{l^2}{2}$$

$$= \frac{YAl^2}{2L}$$

$$= \frac{1}{2}\ x\ \mathrm{Y\ x}\ (\frac{l}{L})^{\ 2}\ \mathrm{x\ A}\ l$$

$$= \frac{1}{2}\ x\ \text{Young's modulus x}\ (srtain)^{\ 2}\ \text{x volume}$$

$$= \frac{1}{2}\ x\ \frac{stress}{strain}\ \mathrm{x}\ (srtain)^{\ 2}\ \text{x volume}$$

$$W = \frac{1}{2}\ \text{x stress x strain x volume}$$

This work done is equal to elastic Potential Energy.

Elastic PE $U = \frac{1}{2}$ x stress x strain x volume

Energy stored per unit volume

$$u = \frac{\text{Energy}}{volume}$$

$u = \frac{1}{2}$ x stress x strain

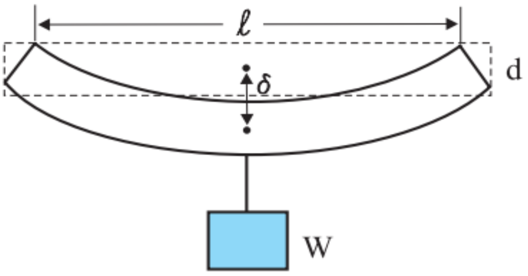
Applications of Elastic Behaviour of Materials

1.Cranes used for lifting and moving heavy loads have a thick metal rope . This is due to the fact that metals have greater youngs modulus.

Also, the elongation of the rope should not exceed the elastic limit. For this thicker rope of radius about 3 cm is recommended. A single wire of this radius would practically be a rigid rod. **So the ropes are always made of a number of thin wires braided together,** like in pigtails, for ease in manufacture, flexibility and strength.

2.The maximum height of a mountain on earth is ~10 km. The height is limited by the elastic properties of rocks.

3.In the construction of bridges and buildings , the beams should not bend too much or break. To reduce the bending for a given load, a material with a large Young's modulus Y is used.



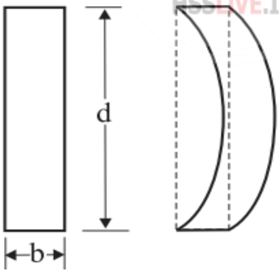
A beam of length l, breadth b, and depth d when loaded at the centre by a load W sags by an amount given by

$$\delta = \frac{W\,l^3}{4bd^3Y}$$

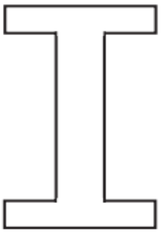
For a given load, the bending reduces when a material with a large Young's modulus Y is used.Bending can also be reduced by increasing thebreadth b, and depth d of the beam.

Buckling

Bending can be effectively reduced by increasing the depth d of the beam. But on increasing the depth, unless the load is exactly at the right place , the deep bar may bend sidewise(as in figure).This is called **buckling.**



To avoid buckling, beams with cross-sectional shape of I is used.



This section provides a large load bearing surface and enough depth to prevent bending. This shape reduces the weight of the beam without sacrificing the strength and hence reduces the cost.

Chapter 9
Mechanical Properties of Fluids

Liquids and gases can flow and are therefore, called fluids.
The fluid does not have any resistance to change of its shape. Thus, the shape of a fluid is governed by the shape of its container.

Basic difference between Liquids and Gases

A liquid is incompressible and has a free surface of its own. A gas is compressible and it expands to occupy all the space available to it. Gas has no free surface.

Pressure

The normal force(F) exerted by a fluid on an area A is called pressure.

$$\text{Pressure, } P = \frac{F}{A}$$

Pressure is a scalar quantity.
Its SI unit is Nm^{-2} or pascal (Pa)
Dimensional formula is $ML^{-1}T^{-2}$

A common unit of pressure is the atmosphere (atm). It is the pressure exerted by the atmosphere at sea level.

$$1\ \text{atm} = 1.013 \times 10^5\ \text{Pa.}$$

Density

Density ρ for a fluid of mass m occupying volume V is given by

$$\rho = \frac{m}{V}$$

It is a positive scalar quantity.
Its SI unit is kg m^{-3}.
The dimensions of density are $[ML^{-3}]$.

The density of water at 4^0C (277 K) is 1000 kg m^{-3}.

A liquid is incompressible and its density is therefore, nearly constant at all pressures. Gases, on the other hand exhibit a large variation in densities with pressure.

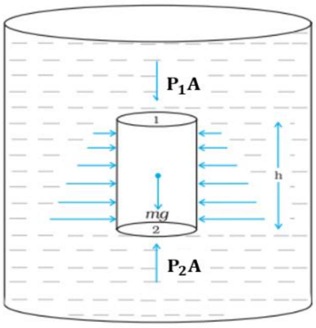
Relative Density

The relative density of a substance is the ratio of its density to the density of water at 4^0C.

$$\text{Relative density} = \frac{\text{Density of substance}}{\text{Density of water at } 4^0\text{C}}$$

It is a dimensionless positive scalar quantity.

Variation of Pressure with Depth



A fluid is at rest in a container. Consider a cylindrical element of fluid having area of base A and height h.
In equilibrium, the resultant vertical forces should be balanced.

$$P_2A = P_1A + mg$$
$$P_2A - P_1A = mg$$
$$(P_2 - P_1)A = mg$$

But $m = \rho V$

$V = hA$

$m = \rho hA$

$$(P_2 - P_1)A = \rho hA\,g$$
$$P_2 - P_1 = \rho\,gh$$

If the point 1 at the top of the fluid , which is open to the atmosphere, P_1 may be replaced by atmospheric pressure (P_a) and we replace P_2 by P

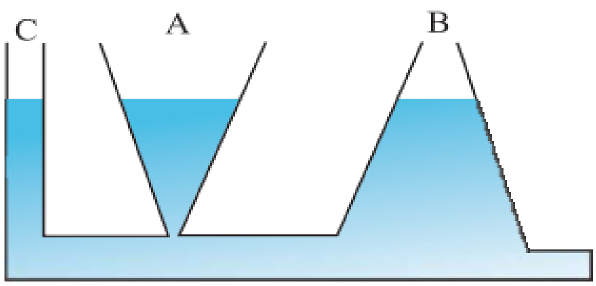
Gauge pressure, $P - P_a = \rho\,gh$

The excess of pressure, $P - P_a$, at depth h is called a gauge pressure at that point.

Absolute Pressure, $P = P_a + \rho\,gh$

Thus, the absolute pressure P, at depth below the surface of a liquid open to the atmosphere is greater than atmospheric pressure by an amount ρgh.

Hydrostatic paradox.



The absolute pressure depends on the height of the fluid column and not on cross sectional or base area or the shape of the container. The liquid pressure is the same at all points at the same horizontal level (same depth). The result is appreciated through the example of **hydrostatic paradox.**

Example

What is the pressure on a swimmer 10 m below the surface of a lake?

$h = 10\text{ m}$

$\rho = 1000\text{ kg m}^{-3}$ Take $g = 10\text{ m s}^{-2}$

$$\begin{aligned} P &= P_a + \rho gh \\ &= 1.01 \times 10^5 + 1000 \times 10 \times 10 \\ &= 1.01 \times 10^5 + 1 \times 10^5 \\ &= 2.01 \times 10^5\text{ Pa} \\ &\approx 2\text{ atm} \end{aligned}$$

(This is a 100% increase in pressure from surface level. At a depth of 1 km the increase in pressure is 100 atm. Submarines are designed to withstand such enormous pressures.)

Atmospheric Pressure

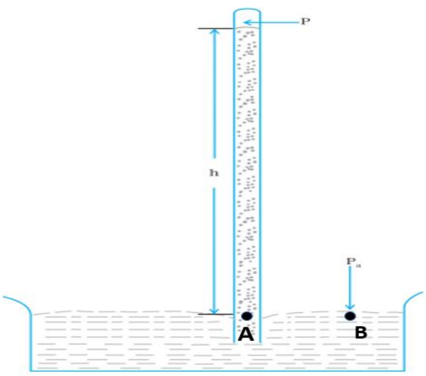
It is the pressure exerted by the atmosphere at sea level.

The pressure of the atmosphere at any point is equal to the weight of a column of air of unit cross sectional area extending from that point to the top of the atmosphere.

$$1\text{ atm} = 1.013 \times 10^5\text{ Pa}$$

Mercury barometer

Mercury barometer is used to measure Atmospheric Pressure. Italian scientist Evangelista Torricelli devised mercury barometer.



(The space above the mercury column in the tube contains only mercury vapour whose pressure P is so small that it may be neglected.)

The pressure inside the column at point A =The pressure at point B, which is at the same level.

Pressure at B = P_a (atmospheric pressure)

Pressure at A = ρgh

$$P_a = \rho gh$$

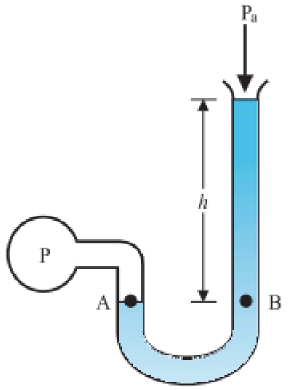
where ρ is the density of mercury and h is the height of the mercury column in the tube.

At sea level h= 76 cm and is equivalent to 1 atm.

The unit of the pressure is the pascal (Pa). It is the same as N m^{-2}. Other common units of pressure are
1 atm = 1.01×10^5 Pa
1 bar = 10^5 Pa
1 torr = 133 Pa = 0.133 kPa
1 mm of Hg = 1 torr = 133 Pa

Open-tube manometer

An open-tube manometer is a used for measuring Guage pressure or pressure differences.



It consists of a U-tube containing a suitable liquid i.e. a low density liquid (such as oil) for measuring small pressure differences and a high density liquid (such as mercury) for large pressure differences.

One end of the tube is open to the atmosphere and other end is connected to the system whose pressure we want to measure .

The pressure at A = pressure at point B

$$P = P_a + \rho gh$$

$$\mathbf{P - P_a = \rho gh}$$

The gauge pressure is proportional to manometer height h.

Example

The density of the atmosphere at sea level is 1.29 kg/m3. Assume that it does not change with altitude. Then how high would the atmosphere extend

$$P_a = \rho gh$$

$$h = \frac{P_a}{\rho g}$$

$$h = \frac{1.01 \times 10^5}{1.29 \times 9.8}$$

$$h = 7989 \text{ m} \approx 8 \text{ km}$$

Example

At a depth of 1000 m in an ocean (a) what is the absolute pressure? (b) What is the gauge pressure? (c) Find the force acting on the window of area 20 cm × 20 cm of a submarine at this depth, the interior of which is maintained at sea-level atmospheric pressure.
(The density of sea water is 1.03×10^3 kg m^{-3}, g = 10m s^{-2})

$$h = 1000 \text{ m} \quad , \quad \rho = 1.03 \times 10^3 \text{ kg m}^{-3}$$

(a) Absolute pressure, $P = P_a + \rho g h$

$$= 1.01 \times 10^5 + 1.03 \times 10^3 \times 10 \times 1000$$
$$= 1.01 \times 10^5 + 103 \times 10^5$$
$$= 104.01 \times 10^5 \text{ Pa}$$
$$\approx 104 \text{ atm}$$

(b) Gauge pressure, $P - P_a = \rho g h$

$$= 1.03 \times 10^3 \times 10 \times 1000$$
$$= 103 \times 10^5 \text{ Pa}$$
$$\approx 103 \text{ atm}$$

(c)The pressure outside the submarine is $P = P_a + \rho g h$ and the pressure inside it is P_a.

Hence, the net pressure acting on the window is gauge pressure, $\rho g h$.
Since the area of the window is $A = 0.04 \text{ m}^2$, the force acting on it is

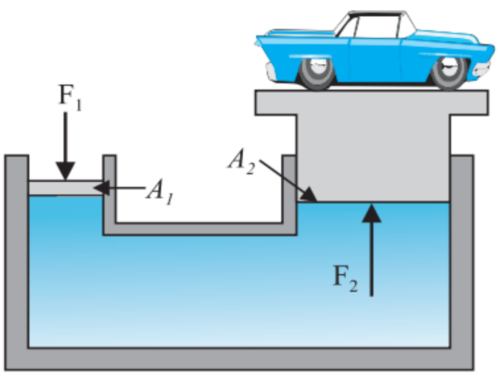
$$F = \text{Gauge Pressure x A}$$
$$= 103 \times 10^5 \times 0.04$$
$$= 4.12 \times 10 \text{ N}$$

Pascal's law for transmission of fluid pressure

Whenever external pressure is applied on any part of a fluid contained in a vessel, it is transmitted undiminished and equally in all directions.

Applications of Pascal's law

1.Hydraulic lift



The pressure on smaller piston

$$P = \frac{F_1}{A_1} \text{--------------}(1)$$

This pressure is transmitted equally to the larger cylinder with a larger piston of area A_2 producing an upward force F_2.

$$P = \frac{F_2}{A_2} \text{--------------}(2)$$

From eq(1) and (2) $\frac{F_1}{A_1} = \frac{F_2}{A_2}$

$$\mathbf{F_2 = F_1 \frac{A_2}{A_1}}$$

Thus, the applied force has been increased by a factor of $\frac{A_2}{A_1}$ and this factor is the mechanical advantage of the device.

Example

Two syringes of different cross sections (without needles) filled with water are connected with a tightly fitted rubber tube filled with water. Diameters of the smaller piston and larger piston are 1.0 cm and 3.0 cm respectively. (a) Find the force exerted on the larger piston when a force of 10 N is applied to the smaller piston. (b) If the smaller piston is pushed in through 6.0 cm, how much does the larger piston move out?

$$F_2 = F_1 \frac{A_2}{A_1}$$

$$F_2 = 10 \text{ x} \frac{\pi \text{ x } (1.5 \text{ x } 10^{-2})^2}{\pi \text{ x } (0.5 \text{ x } 10^{-2})^2}$$

$$= 10 \text{ x}9$$

$$=90 \text{ N}$$

(b) Volume covered by the smaller piston is equal to volume moved by the larger piston.

$$L_1 A_1 = L_2 A_2$$

$$L_2 = L_1 \frac{A_1}{A_2}$$

$$= 6 \text{ x}10^{-2} \text{ x} \frac{\pi \text{ x } (0.5 \text{ x } 10^{-2})^2}{\pi \text{ x } (1.5 \text{ x } 10^{-2})^2} =0.54\text{m}$$

$$= 6 \text{ x}10^{-2} \text{ x}0.111$$

$$= 0.67 \text{ x}10^{-2}\text{m}$$

$$= 0.67\text{cm}$$

Example

In a car lift compressed air exerts a force F_1 on a small piston having a radius of 5.0 cm. This pressure is transmitted to a second piston of radius 15 cm . If the mass of the car to be lifted is 1350 kg, calculate F_1. What is the pressure necessary to accomplish this task? $(g = 9.8\ ms^{-2})$.

$$F_1 = F_2(\frac{A_1}{A_2})$$

$$F_2 = mg = 1350 \times 9.8$$
$$=13230N$$

$$F_1 = 13230 \times \frac{\pi \times (5 \times 10^{-2})^2}{\pi \times (15 \times 10^{-2})^2}$$
$$= 13230 \times \frac{25}{225}$$
$$=1470\ N$$

The air pressure that will produce this force is

$$P = \frac{F_1}{A_1}$$
$$P = \frac{1470}{3{\cdot}14 \times (5 \times 10^{-2})^2}$$
$$=1.9 \times 10^5\ Pa$$

2.Hydraulic brakes

When we apply a force on the pedal with our foot the master piston moves inside the master cylinder, and the pressure caused is transmitted through the brake oil to act on a piston of larger area. A large force acts on the piston and is pushed down expanding the brake shoes against brake lining. In this way a small force on the pedal produces a large retarding force on the wheel.

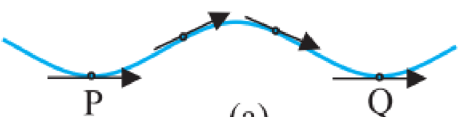
The pressure set up by pressing pedal is transmitted equally to all cylinders attached to the four wheels so that the braking effort is equal on all wheels.

Streamline Flow (Steady Flow)

The study of the fluids in motion is known as fluid dynamics.

The flow of the fluid is said to be steady if at any given point, the velocity of each passing fluid particle remains constant in time.

The velocity of a particular particle may change as it moves from one point to another.



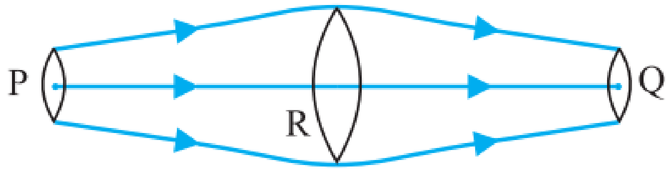
(a)

The path taken by a fluid particle under a steady flow is a streamline.

Streamline is defined as a curve whose tangent at any point is in the direction of the fluid velocity at that point.

No two streamlines can cross, for if they do, an oncoming fluid particle can go either one way or the other and the flow would not be steady.

Equation of Continuity



Consider a region of streamline flow of a fluid. The points P, R and Q are planes perpendicular to the direction of fluid flow . The area of cross-sections at these points are A_P , A_R , A_Q and speeds of fluid particles are v_P, v_R and v_Q.

The mass of fluid crossing at P in a small interval of time $\Delta t = \rho_P A_P v_P \Delta t$

The mass of fluid crossing at Q in a small interval of time $\Delta t = \rho_Q A_Q v_Q \Delta t$

The mass of fluid crossing at R in a small interval of time $\Delta t = \rho_R A_R v_R \Delta t$

The mass of liquid flowing out = The mass of liquid flowing in

$$\rho_P A_P v_P \Delta t = \rho_Q A_Q v_Q \Delta t = \rho_R A_R v_R \Delta t$$

If the fluid is incompressible $\rho_P = \rho_Q = \rho_R$

$$\mathbf{A_P v_P = A_Q v_Q = A_R v_R}$$

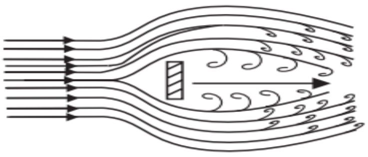
$$\mathbf{Av = constant}$$

This is called the equation of continuity and it is a statement of conservation of mass in flow of incompressible fluids.

Thus, at narrower portions where the streamlines are closely spaced, velocity increases and its vice versa.

Turbulent Flow

Steady flow is achieved at low flow speeds. Beyond a limiting value, called critical speed, the flow of fluid loses steadiness and becomes turbulent.



A jet of air striking a flat plate placed perpendicular to it is an example of turbulent flow.

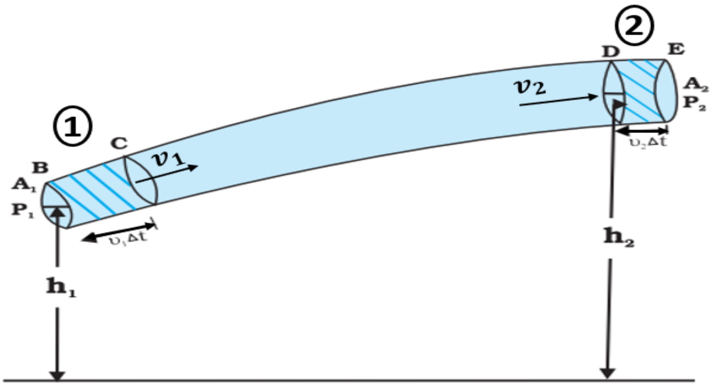
Bernoulli's Principle

Bernoulli's principle states that as we move along a streamline, the sum of the pressure , the kinetic energy per unit volume and the potential energy per unit volume remains a constant.

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

The equation is basically the **conservation of energy applied to non viscous fluid motion in steady state.**

Proof



Consider the flow of an ideal fluid in a pipe of varying cross section, from region **(1)** to region **(2)**. The fluid in the two region is displaced a length of $v_1\ \Delta t$ and $v_2\ \Delta t$ in time Δt.

The work done on the fluid at left end (BC) is

$$W_1 = F_1\Delta x_1 = P_1A_1\,(v_1\Delta t) \qquad | \quad (A_1v_1\Delta t = \Delta V \text{ ,volume})$$

$$W_1 = P_1\Delta V.$$

The work done by the fluid at the end (DE) is

$$W_2 = F_2\Delta x_2 = P_2A_2\,(v_2\Delta t)$$

$$W_2 = P_2\Delta V.$$

(or)The work done on the fluid at the end (DE) is

$$W_2 = -P_2\Delta V.$$

The total work done on the fluid is

$$W_1 + W_2 = P_1\Delta V - P_2\Delta V$$

$$\mathbf{W_1 + W_2 = (P_1 - P_2)\Delta V} \text{-------------}(1)$$

Part of this work goes into changing the kinetic energy of the fluid, and part goes into changing the gravitational potential energy.

The change in its kinetic energy is

$$\Delta K = \frac{1}{2}m\left(v_2^2 - v_1^2\right) \text{ -------------}(2)$$

The change in gravitational potential energy is

$$\Delta U = mg(h_2 - h_1) \text{ ----------------}(3)$$

By work – energy theorem

$$W_1 + W_2 = \Delta K + \Delta U$$

Substituting from eq(1),(2) and (3)

$$(\mathbf{P_1 - P_2})\mathbf{\Delta V} = \frac{1}{2}\mathbf{m}\left(\mathbf{v_2^2 - v_1^2}\right) + \mathbf{mg(h_2 - h_1)} \text{-------------}(4)$$

$m = \rho\Delta V$

$\rho = \frac{m}{\Delta V}$

Divide each term by ΔV to obtain ,

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1)$$

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 + \rho g h_2 - \rho g h_1$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

$$\mathbf{P + \frac{1}{2}\rho v^2 + \rho g h = constant} \text{----------}(5)$$

This is Bernoulli's theorem

When a fluid is at rest i.e. its velocity is zero everywhere, Bernoulli's equation becomes

$$\mathbf{P_1 + \rho g h_1 = P_2 + \rho g h_2}$$

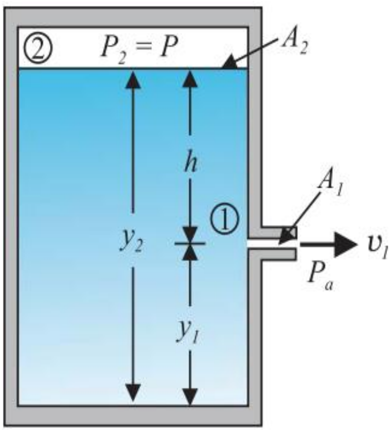
Note:-Bernoulli's theorem is applicable only to the streamline flow of non viscous and incompressible fluids.

Applications of Bernoulli's Principle

1.Speed of Efflux: Torricelli's Law

The word efflux means fluid outflow

Torricelli's law states that the speed of efflux of fluid through a small hole at a depth h of an open tank is equal to the speed of a freely falling body i.e, $\sqrt{2gh}$



Consider a tank containing a liquid of density ρ with a small hole in its side at a height y_1 from the bottom.

According to Bernoulli principle

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Consider regions 1 and 2

According to equation of continuity ,since $(A_2 >> A_1)$, $v_2 = 0$.

$$P_a + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P + \rho g y_2$$

$$\frac{1}{2}\rho v_1^2 = \rho g(y_2 - y_1) + P - P_a$$

$$y_2 - y_1 = h$$

$$\frac{1}{2}\rho v_1^2 = \rho g h + P - P_a$$

$$v_1^2 = 2gh + \frac{2(P - P_a)}{\rho}$$

$$\mathbf{v_1 = \sqrt{2gh + \frac{2(P - P_a)}{\rho}}}$$

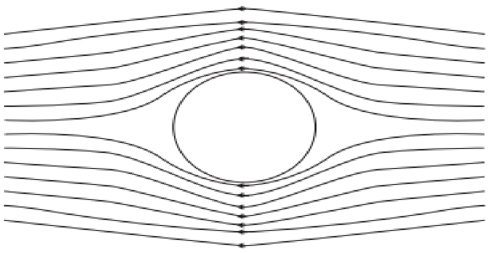
If the tank is open to the atmosphere, then $P = P_a$

$$\mathbf{v_1 = \sqrt{2gh}}$$

This equation is known as Torricelli's law.
This is the speed of a freely falling body.

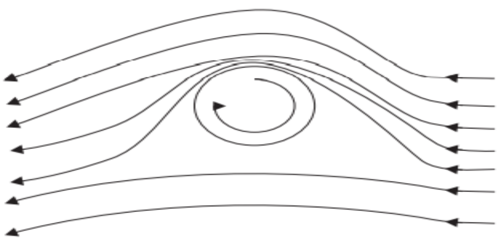
2.Dynamic Lift

(i)Ball moving without spin:



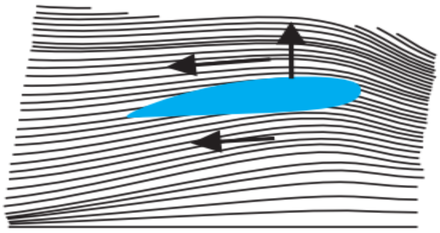
The velocity of fluid (air) above and below the ball at corresponding points is the same resulting in zero pressure difference. The air therefore, exerts no upward or downward force on the ball.

(ii)Ball moving with spin:Magnus Effect



The ball is moving forward and relative to it the air is moving backwards. Therefore, the relative velocity of air above the ball is larger and below it is smaller. This difference in the velocities of air results in the pressure difference between the lower and upper faces and there is a net upward force on the ball. This dynamic lift due to spining is called **Magnus effect.**

(iii)Aerofoil or lift on aircraft wing

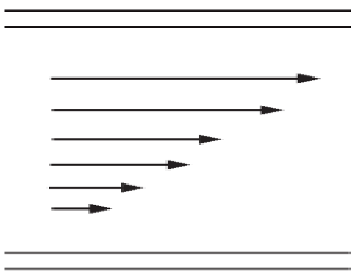


Aerofoil is a solid piece shaped to provide an upward dynamic lift when it moves horizontally through air.

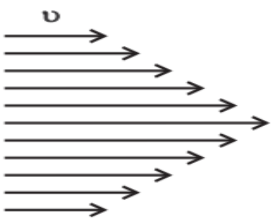
When the aerofoil moves against the wind, the orientation of the wing relative to flow direction causes the streamlines to crowd together above the wing more than those below it. The flow speed on top is higher than that below it. There is an upward force resulting in a dynamic lift of the wings and this balances the weight of the plane.

Viscosity

The internal frictional force that acts when there is relative motion between layers of the liquid is called viscosity.

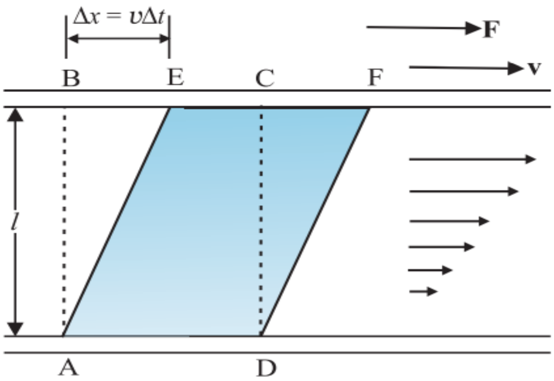


When liquid flows between a fixed and moving glass plates, the layer of the liquid in contact with top surface moves with a velocity v and the layer of the liquid in contact with the fixed surface is stationary. The velocities of layers increase uniformly from bottom to the top layer.



When a fluid is flowing in a pipe or a tube, then velocity of the liquid layer along the axis of the tube is maximum and decreases gradually as we move towards the walls where it becomes zero.

Coefficient of viscosity(η)



Due to viscous force, a portion of liquid, which at some instant has the shape ABCD, take the shape of AEFD after short interval of time (Δt).

$$\text{Shearing stress} = \frac{F}{A}$$

$$\text{Shearing strain} = \frac{\Delta x}{l}$$

$$\text{Strain rate} = \frac{\left(\frac{\Delta x}{l}\right)}{\Delta t} = \frac{\Delta x}{l\,\Delta t} = \frac{v}{l}$$

The coefficient of viscosity(η)for a fluid is defined as the ratio of shearing stress to the strain rate.

$$\eta = \frac{\text{Shearing stress}}{\text{Strain rate}} = \frac{\frac{F}{A}}{\frac{v}{l}}$$

$$\boldsymbol{\eta = \frac{Fl}{vA}}$$

The SI unit of coefficient viscosity is poiseiulle (Pl).
Its other units are N s m^{-2} or Pa s.

The dimensions are $[ML^{-1}T^{-1}]$

Generally thin liquids like water, alcohol etc. are less viscous than thick liquids like coal tar, blood, glycerin etc.

The viscosity of liquids decreases with temperature while it increases in the case of gases.

Stokes' Law

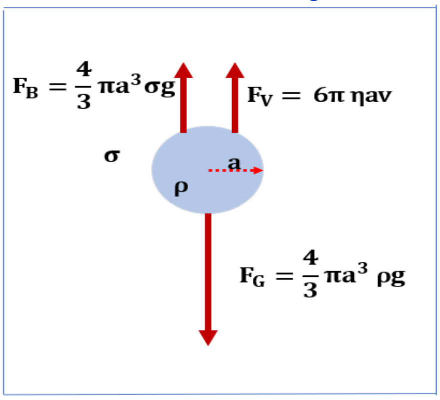
Stokes' law states that the viscous drag force F on a sphere of radius a moving with velocity v through a fluid of coefficient of viscosity η is,

$$\mathbf{F = 6\pi\eta a v}$$

Terminal velocity

When an object falls through a viscous medium (raindrop in air), it accelerates initially due to gravity. As the velocity increases, the retarding force also increases. Finally when viscous force plus buoyant force becomes equal to the force due to gravity(weight of the body), the net force and acceleration become zero. The sphere (raindrop) then descends with a constant velocity clalled terminal velocity.

Expression for Terminal velocity



Consider a raindrop in air. The forces acting on the drop are

1. Force due to gravity(weight,mg) acting downwards , $F_G = \frac{4}{3}\pi a^3 \rho g$
2. Buoyant force acting upwards, $F_B = \frac{4}{3}\pi a^3 \sigma g$
3. Viscous force, $F_V = 6\pi\eta a v$

In equilibrium,

$$6\pi\eta a v + \frac{4}{3}\pi a^3 \sigma g = \frac{4}{3}\pi a^3 g\rho$$

$$6\pi\eta a v = \frac{4}{3}\pi a^3 (\rho - \sigma) g$$

Terminal velocity ,

$$\mathbf{v_t = \frac{2a^2 (\rho - \sigma) g}{9\eta}}$$

So the terminal velocity v_t depends on the square of the radius of the sphere and inversely on the viscosity of the medium.

Example

The terminal velocity of a copper ball of radius 2.0 mm falling through a tank of oil at 20^0C is 6.5 cm s^{-1}. Compute the viscosity of the oil at 20^0C. Density of oil is 1.5×10^3 kg m^{-3}, density of copper is 8.9×10^3 kg m^{-3}.

$v_t = 6.5 \times 10^{-2}\ ms^{-1}$ $\qquad$ $\rho = 8.9 \times 10^3$ kg m^{-3}

$a = 2 \times 10^{-3}$ m $\qquad$ $\sigma = 1.5 \times 10^3$ kg m^{-3}

$g = 9.8 m\ s^{-2}$,

$$v_t = \frac{2a^2\,(\rho-\sigma)g}{9\eta}$$

$$\eta = \frac{2a^2\,(\rho-\sigma)g}{9v_t}$$

$$\eta = \frac{2\ \text{x}\,(2\times10^{-3})^2\,(8.9\times10^3 - 1.5\times10^3)\text{x}9.8}{9\ \text{x}\ 6.5\times10^{-2}}$$

$$\eta = 9.9\times10^{-1}\ \text{kg m}^{-1}\text{s}^{-1}$$

Reynolds Number

Osborne Reynolds defined a dimensionless number, whose value gives one an approximate idea whether the flow would be turbulent . This number is called the Reynolds number(R_e)

$$\mathbf{R_e = \frac{\rho v d}{\eta}}$$

where ρ is the density of the fluid ,v is the speed of fluid, d stands for the dimension of the pipe, and η is the viscosity of the fluid.

R_e <1000 – The flow is streamline or laminar.

$R_e >$ 2000 – The flow is turbulent .

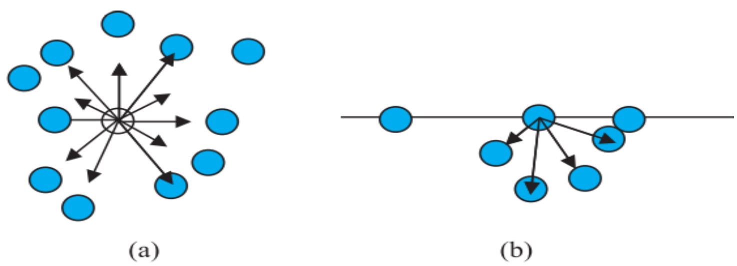
R_e between 1000 and 2000-The flow becomes unsteady .

- **The critical value of Reynolds number at which turbulence sets, is known as critical Reynolds number.** Turbulence dissipates kinetic energy usually in the fm of heat. Racing cars and planes are engineered to precision in order to minimise turbulence.
- Turbulence is sometimes desirable. The blades of a kitchen mixer induce turbulent flow and provide thick milk shakes as well as beat eggs into a uniform texture.

Surface Tension

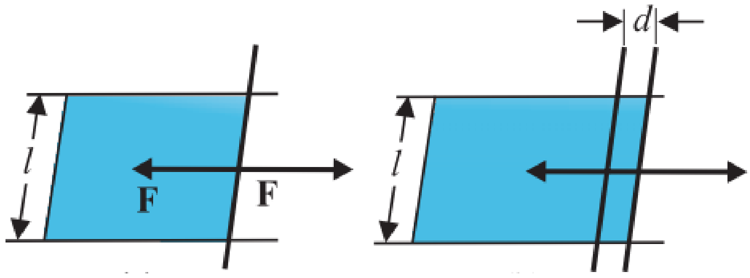
The free surface of a liquid possess some additional energy and it behaves like a stretched elastic membrane. This phenomenon is known as surface tension. Surface tension is concerned with only liquid as gases do not have free surfaces.

Surface Energy



For a molecule well inside a liquid the net force on it is zero. But the molecules on the surface have a net downward pull. So work has to be done against this downward force and this work is stored as energy in suface molecules. Thus, molecules on a liquid surface have some extra energy in comparison to molecules in the interior,which is termed as surface energy. A liquid thus tends to have the least surface area inorder to reduce surface enegy.

Surface Energy and Surface Tension



Consider a horizontal liquid film ending in a movable bar. Due to surface tension the bar is pulled inwards .

Inorder to keep the bar in its original position some work has to be done against this inward full.

$$W = F \times d \text{----------}(1)$$

This work done increases surface energy.

If the surface energy of the film is S per unit area, the extra area is 2d l (film has two sides),

The extra surface energy = S x 2d l ------------(2)

The extra surface energy = work done

$$S \times 2dl = Fd$$

$$\mathbf{S = \frac{F}{2\mathit{l}}}$$

This quantity S is the magnitude of surface tension.

Definition of Surface tension

Surface tension is a force per unit length (or surface energy per unit area) acting in the plane of the interface between the plane of the liquid and any other substance. It is the extra energy that the molecules at the interface have as compared to the interior

$$\text{Surface Tension, } S = \frac{\text{Force}}{\text{Length}}$$

The SI Unit is Nm^{-1}

Dimensional formula is MT^{-2}

The value of surface tension depends on temperature.

The surface tension of a liquid decreases with temperature.

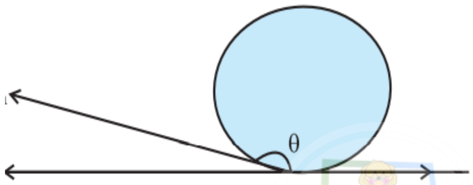
Some effects of surface Tension

- Oil and water do not mix.
- Water wets you and me but not ducks.
- Mercury does not wet glass but water sticks to it.
- Oil rises up a cotton wick, inspite of gravity.
- Sap and water rise up to the top of the leaves of the tree.
- Hairs of a paint brush do not cling together when dry and even when dipped in water but form a fine tip when taken out of it.

Angle of Contact

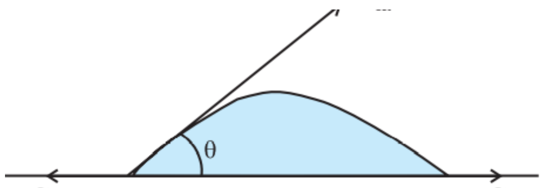
The angle between tangent to the liquid surface at the point of contact and solid surface inside the liquid is termed as angle of contact(θ)
The value of θ determines whether a liquid will spread on the surface of a solid or it will form droplets on it.

When Angle of contact is Obtuse:



When θ is an obtuse angle(greater than 90) then molecules of liquids are attracted strongly to themselves and weakly to those of solid, and liquid then does not wet the solid.
Eg: Water on a waxy or oily surface, Mercury on any surface.

When Angle of contact is Acute:



When θ is an acute angle (less than 90), the molecules of the liquid are strongly attracted to those of the solid and liquid then wets the solid.
Eg: Water on glass or on plastic, Kerosene oil on virtually anything .

Action Soaps and detergents

Soaps, detergents and dying substances are wetting agents. When they are added the angle of contact becomes small so that these may penetrate well and become effective.

Action of Water proofing agents

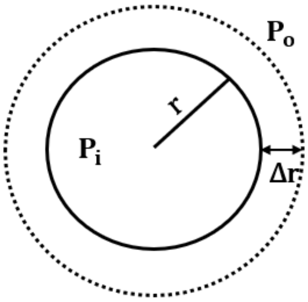
Water proofing agents are added to create a large angle of contact between the water and fibres.

Drops and Bubbles

Why are small drops and bubbles spherical?

Due to surface tension ,liquid surface has the tendency to reduce surface area. For a given volume sphere has minimum surface area. So small drops and bubbles are spherical.
For large drops the effect of gravity predominates that of surface tension and they get flattened.

Excess Pressure inside a spherical drop



Due to surface tension the liquid surface experiences an inward pull and as a result the pressure inside a spherical drop is more than the pressure outside. Due to this excess pressure let the radius of drop increase by Δr

Work done in expansion= Force x Displacement

= Excess pressure x Area x Displacement

$$\mathbf{W = (P_i - P_o) \times 4\pi r^2 \times \Delta r} \text{ ------------------(1)}$$

This workdone is equal to the increase in surface energy

Extra Surface energy = Surface tension x Increase in surface area

Increase in surface area of drop$=4\pi(r+\Delta r)^2 - 4\pi r^2$

$=4\pi\,(r^2 + 2r\Delta r + \Delta r^2 - r^2)$

$=8\pi r\Delta r$ (neglecting higher order terms)

Extra surface energy = S x8πrΔr ------------------(2)

The workdone = extra surface energy

$$\mathbf{(P_i - P_o) \times 4\pi r^2 \times \Delta r = 8\pi r \Delta r\, S} \text{ --------------(3)}$$

$$\mathbf{(P_i - P_o) = \frac{2S}{r}}$$

Excess Pressure Inside a Liquid Bubble

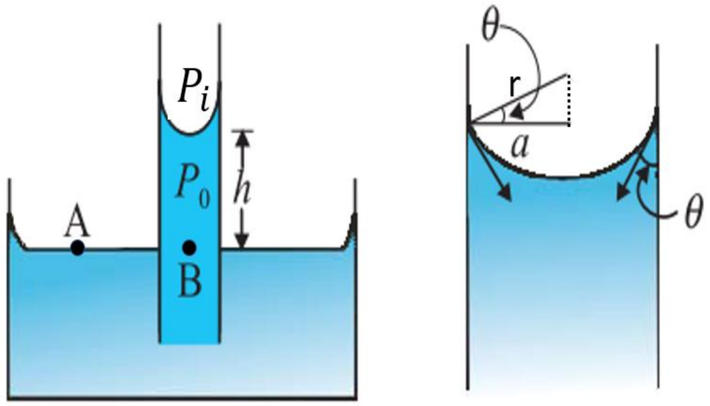
A bubble has two free surfaces.

$$(P_i - P_o) = 2 \times \frac{2S}{r}$$

$$\mathbf{(P_i - P_o) = \frac{4S}{r}}$$

Capillary Rise

Due to the pressure difference across a curved liquid-air interface, water rises up in a narrow tube in spite of gravity. This is called capillary rise.



Consider a vertical capillary tube of circular cross section (radius a) inserted into an open vessel of water.

The excess pressure on the concave meniscus

$$(P_i - P_o) = \frac{2S}{r}$$

$$\cos\theta = \frac{a}{r}$$

$$r = \frac{a}{\cos\theta}$$

$$(P_i - P_o) = \frac{2S}{\frac{a}{\cos\theta}}$$

$$(P_i - P_o) = \frac{2S\cos\theta}{a} \text{----------------}(1)$$

Consider two points A and B in the same horizontal level i.e, the points are at the same pressure.

$$\text{Pressure at A} = P_i$$

$$\text{Pressure at B} = P_o + h\rho g$$

$$P_i = P_o + h\rho g$$

$$P_i - P_o = h\rho g \text{----------------}(2)$$

From eq(1) and (2)

$$h\rho g = \frac{2S\cos\theta}{a}$$

$$\mathbf{h = \frac{2S\cos\theta}{\rho g a}}$$

Thus capillary rise is a consequence of surface tension.
Capillary rise is larger, for capillary tube with smaller radius a.

Note:

If the liquid meniscus is convex, as for mercury, angle of contact θ will be obtuse . Then cos θ is negative and hence value of h will be negative. it is clear that the liquid will lower in the capillary and this is called **capillary fall or capillary depression.**

Example

Find the capillary rise when a capillary tube of radius 0.05 cm is dipped vertically in water. Surface tension for water is $0.073Nm^{-1}$ .Density of water is 1000 kgm^{-3}.

$$h = \frac{2S\cos\theta}{\rho\, ga}$$

For water-glass angle of contact $\theta = 0$, cos 0 =1

$$h = \frac{2S}{\rho\, ga}$$

$$h = \frac{2x\ 0.073}{1000\ x\ 9.8\ x\ 0.05\ x10^{-3}}$$

$$h = 2.98 \times 10^{-2}\ \text{m} = 2.98\ \text{cm}.$$

Chapter 10
Thermal Properties of Matter

Temperature and Heat

Temperature

Temperature is a relative measure, or indication of hotness or coldness.
An object that has a higher temperature than another object is said to be hotter.

SI unit of temperature is kelvin (K).
° C(degree celsius) , ° F(degree fahrenheit) are other commonly used unit of temperature.

Heat

Heat is the form of energy transferred between two systems or a system and its surroundings by virtue of temperature difference.

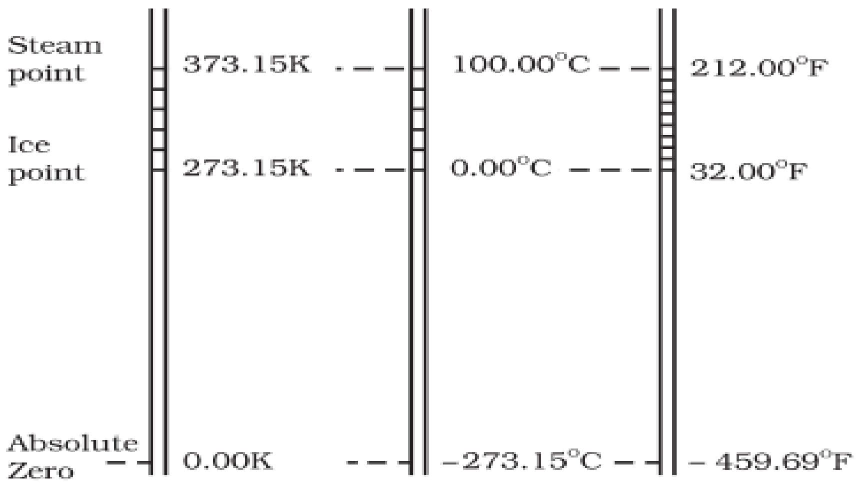
When the temperature of body and its surrounding medium are different, heat transfer takes place between the system and the surrounding medium, until the body and the surrounding medium are at the same temperature.

The SI unit of heat energy is joule (J)

Measurement of Temperature

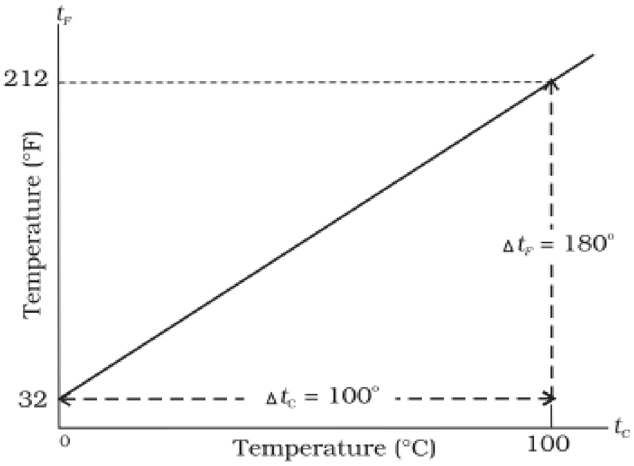
A measure of temperature is obtained using a thermometer.
Variation of the volume of a liquid with temperature is used as the basis for constructing thermometers.
Mercury and alcohol are the liquids used in most liquid-in-glass thermometers.

Comparison of the Kelvin, Celsius and Fahrenheit temperature scales.



- On Fahrenheit scale, there are 180 equal intervals between the ice and steam points.
- On Celsius scale, there are 100 equal intervals between the ice and steam points.
- On Kelvin scale, there are 100 equal intervals between the ice and steam points.

A plot of Fahrenheit temperature (t_F) versus Celsius temperature (t_C).



Temperature on Fahrenheit scale and Celsius scales are related by

$$\frac{t_F - 32}{180} = \frac{t_C}{100}$$

Temperature on Kelvin and Celsius scales are related by

$$T = t_C + 273.15$$

Ideal-Gas Equation and Absolute Temperature

Boyle's law

At constant temperature , the pressure of a quantity of gas is inversely proportional to volume.

$$P \propto \frac{1}{V}$$

$$PV = \text{constant.} \text{ --------------(1)}$$

Charles' law

At constant pressure , the volume of a quantity of gas is directly proportional to temperature .

$$V \propto T$$

$$\frac{V}{T} = \text{constant} \text{------------(2)}$$

Ideal gas law

Low density gases obey Boyle's law and Charles' law , which may be combined into a single relationship.

Combining eq(1) and (2)

$$\frac{PV}{T} = \text{constant}$$

For any quantity of any dilute gas the law can be generalised as

$$\frac{PV}{T} = \mu R$$

$$\mathbf{PV = \mu RT}$$

This is called ideal-gas equation

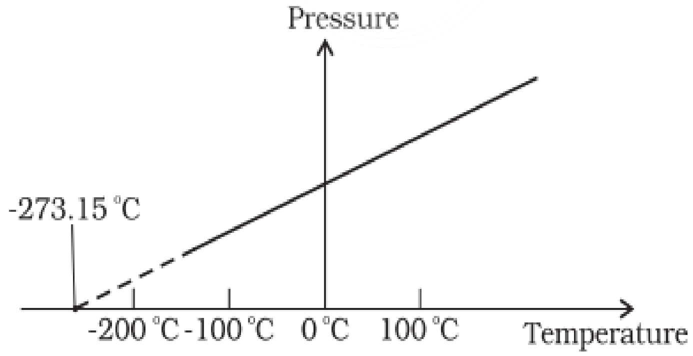
where, μ is the number of moles in the sample of gas.

R is called universal gas constant: $R = 8.31 \text{ J mol}^{-1} \text{ K}^{-1}$

Pressure versus temperature curve of a low density gas kept at constant volume

For ideal gas, $PV = \mu RT$

If volume of a gas is kept constant, it gives $P \propto T$. A plot of pressure versus temperature gives a straight line in this case.



Absolute zero Temperature or Zero kelvin (OK)

The minimum temperature for an ideal gas is called **Absolute temperature or zero kelvin(OK).** This temperature is found to be **– 273.15 °C**

It is obtained by extrapolating the straight line of Pressure – temperature (at constant V) to the axis.

Thermal Expansion

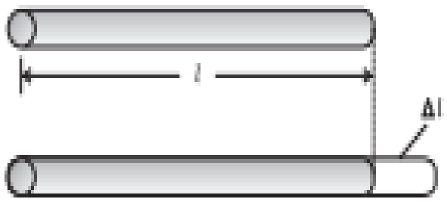
The increase in the dimensions of a body due to the increase in its temperature is called thermal expansion.

Three types of thermal expansions are

1.Linear expansion
2.Area expansion
3.Volume expansion

1.Linear Expansion

The expansion in length is called linear expansion.



If the substance is in the form of a long rod,

The fractional change in length, $\frac{\Delta l}{l} \propto \Delta T.$

$$\frac{\Delta l}{l} = \alpha_l \, \Delta T$$

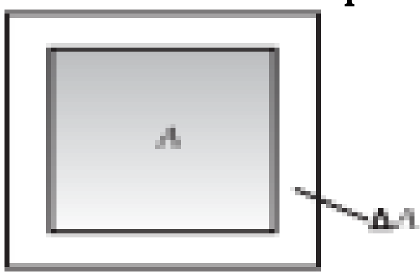
$$\boldsymbol{\alpha_l = \frac{\Delta l}{l \, \Delta T}}$$

where α_l is known as the **coefficient of linear expansion** and is characteristic of the material of the rod.

- Metals expand more and have relatively high values of α_l .
- Copper expands about five times more than glass for the same rise in temperature.

2.Area Expansion

The expansion in area is called area expansion



The fractional change in area, $\frac{\Delta A}{A} \propto \Delta T.$

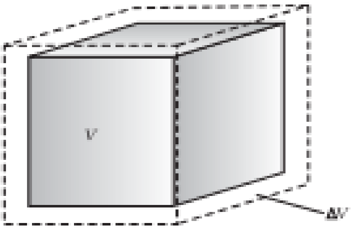
$$\frac{\Delta A}{A} = \alpha_a \Delta T$$

$$\boldsymbol{\alpha_a = \frac{\Delta A}{A \, \Delta T}}$$

where α_a is known as the **coefficient of area expansion .**

3. Volume Expansion

The expansion in volume is called volume expansion



The fractional change in volume, $\frac{\Delta V}{V} \propto \Delta T$

$$\frac{\Delta V}{V} = \alpha_V \Delta T$$

$$\boldsymbol{\alpha_V = \frac{\Delta V}{V \Delta T}}$$

where α_V is known as the **coefficient of volume expansion .**

The value of α_V for alcohol (ethyl) is more than mercury and it expands more than mercury for the same rise in temperature.

Relation between α_l and α_a

$$\alpha_a = \frac{\Delta A}{A\,\Delta T}$$

$$\Delta A = (l + \Delta l)^2 - l^2$$

$$\Delta A = 2\,l\,\Delta\,l \qquad \text{(Neglecting term } (\Delta l)^2 \text{)}$$

$$A = l^2$$

$$\alpha_a = \frac{2\,l\,\Delta l}{l^2 \Delta T}$$

$$\alpha_a = 2\,\frac{\Delta l}{l\,\Delta T}$$

$$\frac{\Delta l}{l\,\Delta T} = \alpha_l$$

$$\boldsymbol{\alpha_a = 2\,\alpha_l} \text{ ---------(1)}$$

Relation between α_l and α_v

$$\alpha_V = \frac{\Delta V}{V \Delta T}$$

$$\Delta V = (l + \Delta l)^3 - l^3$$

$$\Delta V = 3\,l^2\,\Delta l \qquad \text{(Neglecting terms } (\Delta l)^2 \text{and } (\Delta l)^3\text{)}$$

$$V = l^3$$

$$\alpha_v = \frac{3\,l^2\,\Delta l}{l^3 \Delta T}$$

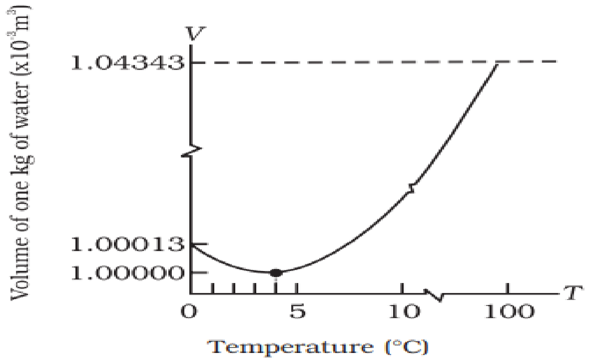
$$\alpha_v = 3\,\frac{\Delta l}{l\,\Delta T}$$

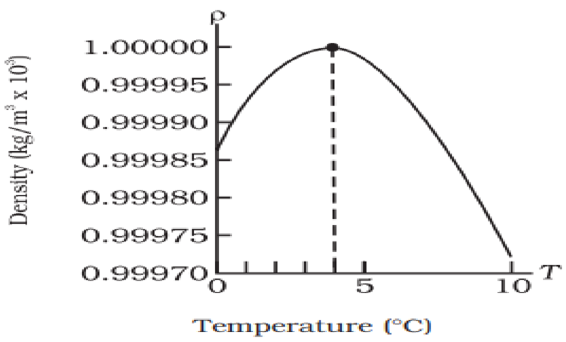
$$\frac{\Delta l}{l\,\Delta T} = \alpha_l$$

$$\boldsymbol{\alpha_v = 3\,\alpha_l} \text{ -----------(2)}$$

From eqs(1) and (2) $\boldsymbol{\alpha_l : \alpha_a : \alpha_v = 1 : 2 : 3}$

Thermal Expansion of Water(Or) Anomalous Behaviour of Water





Water exhibits an anomalous behaviour; it contracts on heating from 0 °C to 4°C.When it is heated after 4°C ,it expands like other liquids. This means that **water has minimum volume and hence maximum density at 4 °C .**

Why the bodies of water, such as lakes and ponds, freeze at the top first?

This is due to anomalous expansion of water. As a lake cools toward 4 °C, water near the surface becomes denser, and sinks. Then the warmer, less dense water near the bottom rises. When this layer cools below 4 °C, it freezes, and being less dense, remain at the surfaces. Thus water bodies freeze at the top first. Water at the bottom protects aquatic animal and plant life.

The coefficient of volume expansion at constant pressure for an ideal gas

The ideal gas equation, $PV = \mu RT$ ------------------(1)

At constant pressure, $P\Delta V = \mu R\,\Delta T$

$$\frac{\Delta V}{V} = \frac{\mu R\,\Delta T}{P\,V}$$

From eq(1), $\frac{\mu R}{P\,V} = \frac{1}{T}$

$$\frac{\Delta V}{V} = \frac{\Delta T}{T}$$

$$\frac{\Delta V}{V\Delta T} = \frac{1}{T}$$

$$\alpha_v = \frac{1}{T}$$ (for an ideal gas at constant pressure)

Thermal Stress

If the thermal expansion of a rod is prevented by fixing its ends rigidly, the rod acquires a compressive strain. The corresponding stress set up in the rod is called thermal stress.

Heat Capacity

Heat capacity (S) of a substance is the amount of heat required to raise the temperature of the substance by one unit.

$$\mathbf{S = \frac{\Delta Q}{\Delta T}}$$

Unit is JK^{-1}

Heat capacity of a substance depends on its mass, temperature and the nature of substance.

Specific Heat capacity

Specific heat capacity (s) of a substance is the amount of heat required to raise the temperature of unit mass of the substance by one unit.

$$\text{Specific heat capacity} = \frac{\text{Heat capacity}}{\text{mass}}$$

$$s = \frac{S}{m}$$

$$\text{but,} \quad S = \frac{\Delta Q}{\Delta T}$$

$$\mathbf{s = \frac{1}{m} \frac{\Delta Q}{\Delta T}}$$

Unit is $Jkg^{-1}K^{-1}$

It depends on the nature of the substance and its temperature. It is independent of mass of the substance.

From above equation, the amount of heat,

$$\mathbf{\Delta Q = m\, s\, \Delta T}$$

Molar Specific Heat Capacity

Molar Specific heat capacity (C) of a substance is the amount of heat required to raise the temperature of one mole of the substance by one unit.

$$C = \frac{S}{m}$$

$$\mathbf{C = \frac{1}{\mu} \frac{\Delta Q}{\Delta T}}$$

Unit is $Jmol^{-1}K^{-1}$

It depends on the nature of the substance and its temperature. It is independent of mass of the substance.

Specific Heat Capacities of Gases

As gas is compressible, heat transfer can be achieved by keeping either pressure or volume constant. So gases have two types of molar specific heat capacities.

Molar specific heat capacity at constant pressure C_p and

Molar specific heat capacity at constant volume C_v

Molar Specific Heat Capacity at Constant Pressure C_p

Molar specific heat capacity at constant pressure of a substance is the amount of heat required to raise the temperature of one mole of the substance by one unit keeping its pressure constant.

$$C_p = \frac{1}{\mu}\left(\frac{\Delta Q}{\Delta T}\right)_p$$

Molar specific heat capacity at constant volume C_v

Molar specific heat capacity at constant volume of a substance is the amount of heat required to raise the temperature of one mole of the substance by one unit keeping its volume constant.

$$C_v = \frac{1}{\mu}\left(\frac{\Delta Q}{\Delta T}\right)_v$$

Water has the highest specific heat capacity compared to other substances. Specific heat capacity of water is 4186 is $Jkg^{-1}K^{-1}$

- For this reason water is used as a coolant in automobile radiators as well as a heater in hot water bags.
- Owing to its high specific heat capacity, the water warms up much more slowly than the land during summer and consequently wind from the sea has a cooling effect.

In desert areas, the earth surface warms up quickly during the day and cools quickly at night.

Calorimetry: **Calorimetry means measurement of heat.**

Calorimeter: **A device in which heat measurement can be made is called a calorimeter.**

It consists a metallic vessel and stirrer of the same material like copper or alumiunium. The vessel is kept inside a wooden jacket which contains heat insulating materials like glass wool etc. The outer jacket acts as a heat shield and reduces the heat loss from the inner vessel. There is an opening in the outer jacket through which a mercury thermometer can be inserted into the calorimeter.

Change of State

Matter normally exists in three states: solid, liquid, and gas.

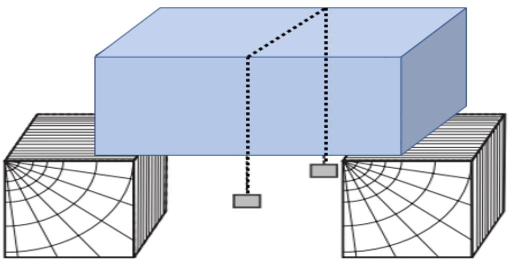
A transition from one of these states to another is called a change of state.
The temperature of the system does not change during change of state.

Change of state from solid to liquid

The change of state from solid to liquid is called melting and from liquid to solid is called fusion.

- Both the solid and liquid states of the substance coexist in thermal equilibrium during the change of states from solid to liquid.
- **The temperature at which the solid and the liquid states of the substance in thermal equilibrium with each other is called its melting point.**
- **Melting point decrease with increase in pressure** . The melting point of a substance at standard atomspheric pressure is called its normal melting point.

Regelation



When the wire passes through the ice slab , ice melts at lower temperature due to increase in pressure. When the wire has passed, water above the wire freezes again. This phenomenon of refreezing is called **regelation.**

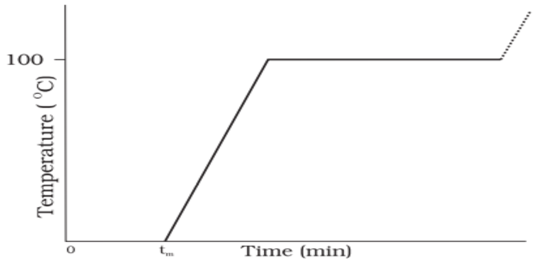
Skating is possible on snow due to the formation of water below the skates. Water is formed due to the increase of pressure and it acts as a lubricant.

The change of state from liquid to vapour

The change of state from liquid to vapour (or gas) is called vaporisation and from vapour to liquid is called condensation.

- The temperature remains constant until the entire amount of the liquid is converted into vapour
- **The temperature at which the liquid and the vapour states of the substance coexist is called its boiling point.**
- **The boiling point increases with increase in pressure and decreases with decreases in pressure.** The boiling point of a substance at standard atmospheric pressure is called its normal boiling point.
  - Cooking is difficult on hills. At high altitudes, atmospheric pressure is lower, boiling point of water decreases as compared to that at sea level.
  - Boiling point is increased inside a pressure cooker by increasing the pressure. Hence cooking is faster.

A plot of temperature versus time showing the changes in the state of ice on heating.



Sublimation

The change from solid state to vapour state without passing through the liquid state is called sublimation, and the substance is said to sublime.
Eg: Dry ice (solid CO2) , Iodine, Camphor

During the sublimation process both the solid and vapour states of a substance coexist in thermal equilibrium.

Change of state	
Solid to Liquid	Melting
Liquid to Solid	Fusion
Liquid to Gas	Vaporisation
Gas to Liquid	Condensation
Solid to Gas	Sublimation

Latent Heat

The amount of heat per unit mass transferred during change of state of the substance is called latent heat of the substance for the process.
The heat required during a change of state depends upon the heat of transformation and the mass of the substance undergoing a change of state.

$$Q = mL$$

$$L = \frac{Q}{m}$$

where L is known as latent heat and is a characteristic of the substance.

SI unit of Latent Heat is J kg^{-1}

The value of L also depends on the pressure. Its value is usually quoted at standard atmospheric pressure

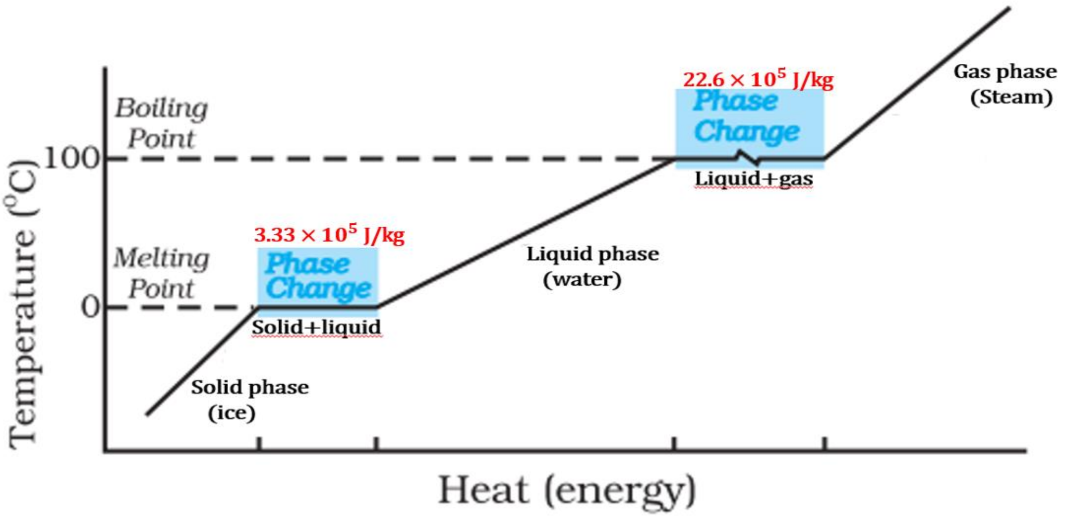
Latent Heat of Fusion (L_f)

The latent heat for a solid -liquid state change is called the latent heat of fusion (L_f) or simply heat of fusion.

Latent Heat of Vaporisation (L_v)

The latent heat for a liquid-gas state change is called the latent heat of vaporisation (L_v) or heat of vaporisation.

Temperature versus heat for water at 1 atm pressure (not to scale).



The slopes of the phase lines are not same, which indicate that specific heats of the various states are not equal.
When slope of graph is less, it indicates a high specific heat capacity .

- The specific heat capacity of water is greater than that of ice.

$$\Delta Q = m\, s\, \Delta T$$

  The amount of heat required ,ΔQ in liquid phase will be greater than that in solid phase for same ΔT.
  So slope of liquid phase is less than that of solid phase.
- For water, the latent heat of fusion is $\mathbf{L_f} = 3.33 \times 10^5$ J kg^{-1}.
  That is 3.33×10^5 J of heat are needed to melt 1 kg of ice at 0 °C.
  For water, the latent heat of vaporisation is $\mathbf{L_v} = 22.6 \times 105$ J kg^{-1}.
  That is 22.6×10^5 J of heat is needed to convert 1 kg of water to steam at 100 °C.

Why burns from steam are usually more serious than those from boiling water?

For water, the latent heat of vaporisation is $\mathbf{L_v} = 22.6 \times 10^5$ J kg^{-1}.
That is 22.6×10^5 J of heat is needed to convert 1 kg of water to steam at 100 °C. So, steam at 100 °C carries 22.6×10^5J kg^{-1}more heat than water at 100 °C. This is why burns from steam are usually more serious than those from boiling water.

Example

When 0.15 kg of ice at 0 °C is mixed with 0.30 kg of water at 50°C in a container, the resulting temperature is 6.7 °C. Calculate the heat of fusion of ice. (s_{water}= 4186 J $kg^{-1}\ K^{-1}$)

$$\Delta Q = m\, s\, \Delta T$$

Heat lost by water $= m s_{water}\ (T_f - T_i)$

$= 0.30 \times 4186 \times (50.0\ °C - 6.7\ °C) = 54376.14$ J

Heat required to melt ice $= m\, L_f$

$= (0.15\ kg)\ L_f$

Heat required to raise temperature of ice water to final temperature

$= m\, s_{water}\ (T_f - T_i)$

$= 0.15 \times 4186\ \times (6.7\ °C - 0\ °C) = 4206.93$ J

Heat lost = heat gained

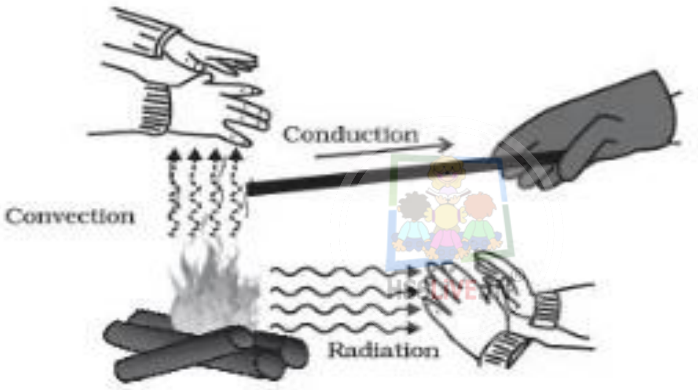
$54376.14\ J = (0.15\ kg)\ L_f + 4206.93\ J$

$$\mathbf{L_f = 3.34 \times 105\ J\ kg^{-1}}$$

Heat Transfer

There are three distinct modes of heat transfer :

conduction, convection and radiation

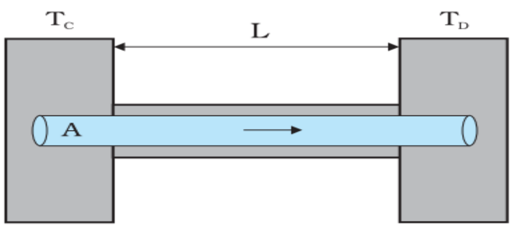


1.Conduction

Conduction is the mechanism of transfer of heat between two adjacent parts of a body because of their temperature difference.

If one end of a metallic rod is heated , heat transfer takes place by conduction from the hot end of the rod to the other end.

Consider a metallic bar of length L and uniform cross section A with its two ends maintained at different temperatures T_C and T_D; ($T_C > T_D$).



The rate of flow of heat H is proportional to the temperature difference (T_C - T_D) and the area of cross section A and is inversely proportional to the length L :

$$H = K A \frac{T_C - T_D}{L}$$

The constant of proportionality K is called the **thermal conductivity** of the material. The greater the value of K for a material, the more rapidly will it conduct heat. The SI unit of K is $Js^{-1}m^{-1}K^{-1}$ or $Wm^{-1}K^{-1}$

- Gases are poor thermal conductors while liquids have conductivities intermediate between solids and gases.
- Metals are good thermal conductors.
- Wood , glass and wool have small thermal conductivities.
- Some cooking pots have copper coating on the bottom. Being a good conductor of heat, copper promotes the distribution of heat over the bottom of a pot for uniform cooking.
- Plastic foams, on the other hand, are good insulators, mainly because they contain pockets of air.
- Houses made of concrete roofs get very hot during summer days, because thermal conductivity of concrete is moderately high. Therefore, people usually prefer to give a layer of earth or foam insulation on the ceiling so that heat transfer is prohibited and keeps the room cooler.

2.Convection

Convection is a mode of heat transfer by actual motion of matter. It is possible only in fluids.

Convection can be natural or forced.

Natural Convection

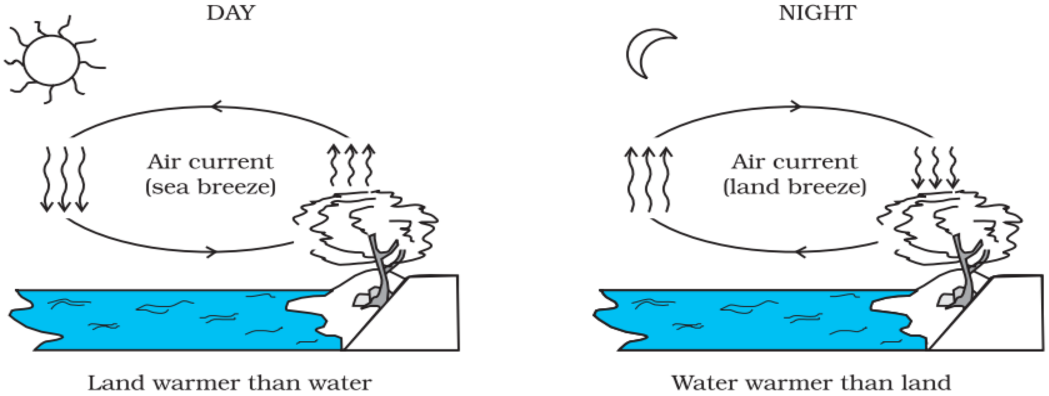
In natural convection, gravity plays an important part. When a fluid is heated from below, the hot part expands and, therefore, becomes less dense. Because of buoyancy, it rises and the upper colder part replaces it. This again gets heated, rises up and is replaced by the colder part of the fluid.**Eg: Sea breeze, Land breeze, Trade wind**

1.Sea breeze

During the day, the ground heats up more quickly than large water bodies. This is due to greater specific heat capacity of water. The air in contact with the warm ground is heated . It expands, becomes less dense and rises . Then cold air above sea moves to fill this space and is called as sea breeze .

2.Land breeze

At night, the ground loses its heat more quickly, and the water surface is warmer than the land. The air in contact with water is heated. It expands, becomes less dense and rises . Then cold air above the ground moves to fill this space and is called as land breeze .



3.Trade wind

The surface of the earth at the equator is heated more by sun rays than poles. The hot air at equator expands, becomes less dense and rises . Then cold air from poles moves to the equator. This is called trade wind.

Forced Convection

In forced convection, material is forced to move by a pump or by some other physical means.

Eg: Forced-air heating systems in home
The human circulatory system
The cooling system of an automobile engine.

In the human body, the heart acts as the pump that circulates blood through different parts of the body, transferring heat by forced convection and maintaining it at a uniform temperature.

3.Radiation

The mechanism for heat transfer which does not require a medium is called radiation.

The electromagnetic radiation emitted by a body by virtue of its temperature is called thermal radiation. The energy so radiated by electromagnetic waves is called radiant energy. All bodies emit radiant energy, whether they are solid, liquid or gases.

Heat is transferred to the earth from the sun through empty space as radiation.

Black bodies absorb and emit radiant energy better than bodies of lighter colours.

This fact finds many applications in our daily life.

- We wear white or light coloured clothes in summer so that they absorb the least heat from the sun.
- During winter, we use dark coloured clothes which absorb heat from the sun and keep our body warm.
- The bottoms of the utensils for cooking food are blackened so that they absorb maximum heat from the fire and give it to the vegetables to be cooked.

Principle of Thermo Bottles

Thermos bottle is a device to minimise heat transfer between the contents of the bottle and outside. It consists of a double-walled glass vessel with the inner and outer walls coated with silver. Radiation from the inner wall is reflected back into the contents of the bottle. The outer wall similarly reflects back any incoming radiation. The space between the walls is evacuated to reduce conduction and convection losses and the flask is supported on an insulator like cork. The device is, therefore, useful for preventing hot contents from getting cold, or alternatively to store cold contents (like ice).

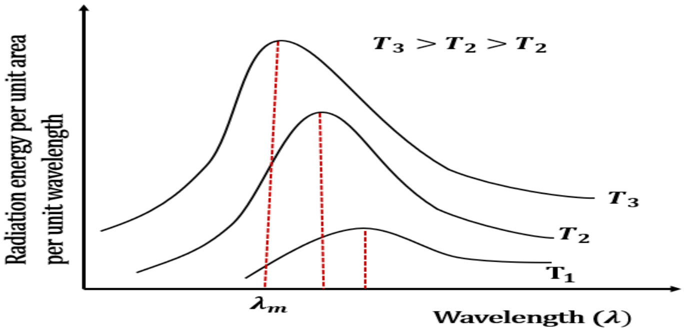
Blackbody

An object that absorbs all radiations falling on it at all wavelength is called a blackbody. A blackbody, also emits radiations of all possible wavelength.

Blackbody Radiation

The radiations emitted by blackbody are called blackbody radiations.

The variation of energy emitted by a blackbody with wavelength is is as shown in figure:



Wien's Displacement Law

The wavelength λ_m for which energy emitted by a blackbody is the maximum , is inversely proportional to the temperarure.This is known as Wien's Displacement Law. $\lambda_m \ \alpha \ \frac{1}{T}$

$$\lambda_m T = \text{constant}$$

The constant is called Wien's constant and its value is 2.9 x 10^{-3}mK.

Stefan-Boltzmann Law

The energy emitted by a perfect radiator (black body) per unit time is given by $\mathbf{H = A\sigma T^4}$

This is called Stefan-Boltzmann Law.

where A - the area of the body

T - temperature of body

σ is called **Stefan-Boltzmann constant**

$\sigma = 5.67 \times 10^{-8} Wm^{-2}K^{-4}$

- If the body is not a perfect radiator

$$\mathbf{H = Ae\sigma T^4}$$

where e is the emissivity i.e, the ability to radiate.

For a perfect radiator,emissivity, e=1

- If T is the temperature of the body and T_s is the temperature of surroundings, then rate of loss of radiant energy is

$$\mathbf{H = Ae\sigma\,(T - T_s)^4}$$

Newton's Law of Cooling

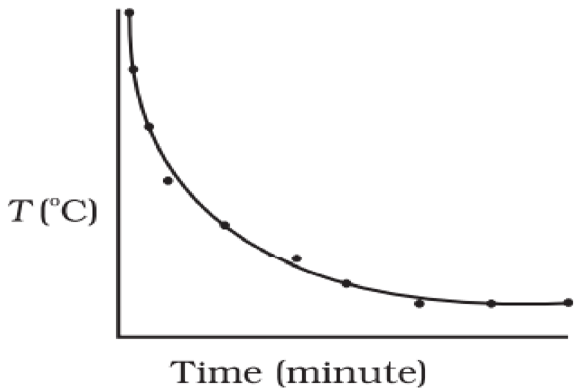
Newton's Law of Cooling says that the rate of loss of heat(rate of cooling) of a body is proportional the difference of temperature of the body and the surroundings. $-\frac{dQ}{dT} = k(T_2 - T_1)$

Where T_1 is the temperature of the surrounding medium

T_2 is the temperature of the body

k is a positive constant depending upon the area and nature of the surface of the body

Curve showing cooling of hot water with time.



Chapter 11
Thermodynamics

Thermodynamics

Thermodynamics is a branch of physics which deals with the study of heat, temperature and their inter conversion of heat energy into other forms of energy.

Thermodynamic System

A thermodynamic system is a certain quantity of matter which is separated from its surroundings by a real or imaginary boundary. This system may be in solid , liquid or gaseous state.

Surroundings

Everything outside a thermodynamic system is its surrounding.

Boundary

The real or imaginary surface that separates the system from its surroundings is called boundary.

Boundary can be of two types: Adiabatic , Diathermic

Adiabatic wall (boundary)

An insulating wall that does not allow flow of energy (heat) from one system to another is called an adiabatic wall.

Diathermic wall

A conducting wall that allows energy flow (heat) from one system to another is called a diathermic wall.

Thermodynamic State Variables

Every equilibrium state of a thermodynamic system is completely described by specific values of some macroscopic variables. These are called thermodynamic state variables.

Eg: pressure, volume, temperature, mass , composition,Entropy, Enthalpy

Thermodynamic Equilibrium

The state of a thermodynamic system is in equilibrium , if the macroscopic variables that characterise the system do not change in time.

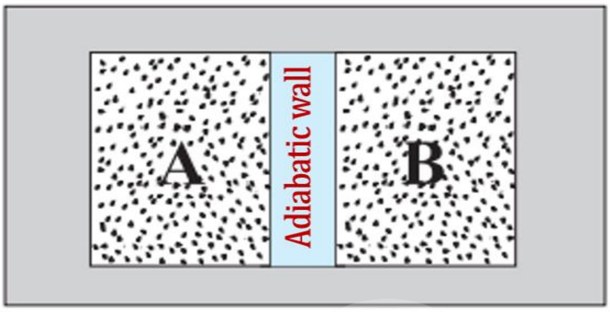
Eg: A gas inside a closed rigid container, completely insulated from its surroundings, with fixed values of pressure, volume, temperature, mass and composition that do not change with time, is in a state of thermodynamic equilibrium.

Thermal equilibrium

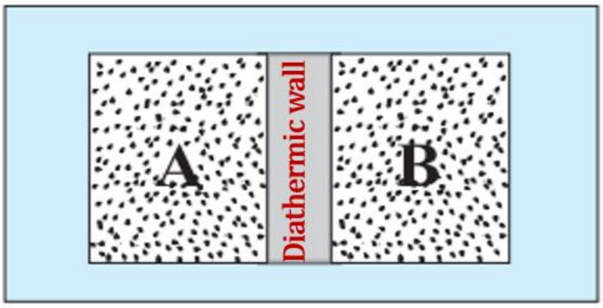
A system is said to be in thermal equilibrium with itself if the temperature of the system remains constant.

Two systems are said to be in thermal equilibrium, when there is no flow of thermal energy between them ,when they are connected by a diathermic wall. In thermal equilibrium, the temperatures of the two systems are equal.

If systems A and B (two gases) are separated by an adiabatic wall that does not allow flow of heat, system A will be in its own thermal equilibrium, and system B also will be in its own thermal equilibrium. But systems A and B will not be in thermal equilibrium with each other.

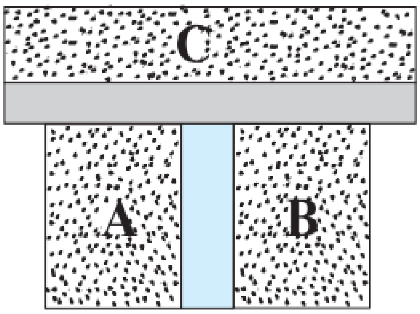


If the same systems A and B separated by a diathermic wall that allows heat to flow from one to another, thermal equilibrium is attained between the two systems in due course. In this case the temperature of the two systems become equal.



Zeroth Law of Thermodynamics

R.H. Fowler formulated this law in 1931 long after the first and second Laws of thermodynamics were stated and so numbered.



Systems A and B are separated by an adiabatic wall, while each is in contact with a third system C via a conducting wall. In this case A will be in thermal equilibrium with B.

Zeroth Law of Thermodynamics states that 'two systems in thermal equilibrium with a third system separately are in thermal equilibrium with each other'.

Thus if A and B are separately in equilibrium with C, $T_A = T_C$ and $T_B = T_C$. This implies that $T_A = T_B$ i.e. the systems A and B are also in thermal equilibrium.

i. e, If $T_A = T_C$ and $T_B = T_C$ then $T_A = T_B$

The concept of temperature from Zeroth Law

The thermodynamic variable whose value is equal for two systems in thermal equilibrium is called temperature (T).

Heat, Internal Energy and Work

Heat and work are two modes of energy transfer to the system . (or)
Heat and work are two modes to change the internal energy of a system.
Heat and work are not thermodynamic state variables, but internal energy is a thermodynamic state variable.

Internal Energy(U)

Internal energy of a system is the sum of kinetic energies and potential energies of the molecular constituents of the system.
It does not include the over-all kinetic energy of the system.

Internal energy U of a system is an example of a thermodynamic 'state variable' – its value depends only on the given state of the system, not on the 'path' taken to arrive at that state.

Heat(Q)

Heat is energy transfer due to temperature difference between two systems.
Heat is certainly energy, but it is the energy in transit.

Heat is not a thermodynamic state variable. its value depends on the 'path' taken to arrive a particular state.

Work (W)

Work is energy transfer brought about by means that do not involve such a temperature difference(e.g. moving the piston by raising or lowering some weight connected to it)

Work is not a thermodynamic state variable. its value depends on the 'path' taken to arrive a particular state.

First Law of Thermodynamics

The heat supplied supplied to the system is partly used to increase the internal energy of the system and the rest is used to do work on the environment .

$$\Delta Q = \Delta U + \Delta W$$

ΔQ = Heat supplied to the system by the surroundings
ΔW = Work done by the system on the surroundings
ΔU = Change in internal energy of the system

- If work is done by the system against a constant pressure P ,then

$$\Delta W = P\,\Delta V \qquad (\Delta W = F\,\Delta x = PA\,\Delta x = P\,\Delta V)$$

$$\Delta Q = \Delta U + P\,\Delta V$$

- If a system is taken through a process in which $\Delta U = 0$

$$\Delta Q = \Delta W$$

Heat supplied to the system is used up entirely by the system in doing work on the environment.

Eg: Isothermal expansion of an ideal gas

- From the First Law of Thermodynamics,

$$\Delta Q - \Delta W = \Delta U$$

As ΔU is path independent ($\Delta Q - \Delta W$)should also be path independent.

i.e., eventhough ΔQ and ΔW are path dependent, their combination $\Delta Q - \Delta W$, is path independent.

Specific Heat Capacities of Gases

As gas is compressible, heat transfer can be achieved by keeping either pressure or volume constant. So gases have two types of molar specific heat capacities.

Molar specific heat capacity at constant pressure C_p

$$C_p = \frac{1}{\mu}\left(\frac{\Delta Q}{\Delta T}\right)_p$$

Molar specific heat capacity at constant volume C_v

$$C_v = \frac{1}{\mu}\left(\frac{\Delta Q}{\Delta T}\right)_v$$

Relation connecting C_p and C_v – Mayer's relation

First law of thermodynamics , $\Delta Q = \Delta U + P\,\Delta V$

If ΔQ is absorbed at constant volume, $\Delta V = 0$

$$\Delta Q = \Delta U$$

$$C_v = \left(\frac{\Delta Q}{\Delta T}\right)_v \qquad \text{(for 1 mole)}$$

$$C_v = \left(\frac{\Delta U}{\Delta T}\right)_v \qquad \text{(U depends only on T .So subscript V can be omitted)}$$

$$C_v = \frac{\Delta U}{\Delta T} \text{---------}(1)$$

If, on the other hand, ΔQ is absorbed at constant pressure,

$$\Delta Q = \Delta U + P\,\Delta V$$

$$C_p = \left(\frac{\Delta Q}{\Delta T}\right)_p$$

$$C_p = \left(\frac{\Delta U}{\Delta T}\right)_p + \left(P\frac{\Delta V}{\Delta T}\right)_p \qquad \text{(U depends only on T .So subscript P can be omitted)}$$

$$C_p = \frac{\Delta U}{\Delta T} + \left(P\frac{\Delta V}{\Delta T}\right)_p \text{------------}(2)$$

Now, for a mole of an ideal gas

$$PV = RT$$

$$P\left(\frac{\Delta V}{\Delta T}\right)_p = R\left(\frac{\Delta T}{\Delta T}\right)$$

$$P\left(\frac{\Delta V}{\Delta T}\right)_p = R$$

$$C_p = \frac{\Delta U}{\Delta T} + R$$

Substituting from eq(1) $\quad C_p = C_v + R$

$$C_p - C_v = R$$

This is called Mayer's relation.

C_p is always greater than C_v . Why

From eqns (1) and (2) $$C_v = \frac{\Delta U}{\Delta T}$$

$$C_p = \frac{\Delta U}{\Delta T} + \left(P\frac{\Delta V}{\Delta T}\right)_p$$

When gas is heated at constant volume, the entire heat is used to increase the internal energy of the gas. But when the gas is heated at constant pressure, the heat is used to increase the internal energy and also to do external work during expansion. $\frac{\Delta U}{\Delta T}$ is the same in both cases. Hence C_p is greater than C_v.

Thermodynamic State Variables and Equation of State

Every equilibrium state of a thermodynamic system is completely described by specific values of some macroscopic variables, also called **state variables**.

For example, an equilibrium state of a gas is completely specified by the values of pressure, volume, temperature, and mass (and composition if there is a mixture of gases).

Equation of state

The connection between the state variables is called the equation of state.

Eg: For an ideal gas, the equation of state is the ideal gas relation

$$PV = \mu RT$$

Extensive and Intensive Variables

The thermodynamic state variables are of two kinds:

Extensive and Intensive.

Extensive Variables

Extensive variables indicate the 'size' of the system.

(If we imagine ,to divide a system in equilibrium into two equal parts, the variables whose values get halved in each part are extensive.)

Eg:Internal energy, Volume , Mass

Intensive Variables

Intensive variables do not indicate the 'size' of the system.

(If we imagine, to divide a system in equilibrium into two equal parts, the variables that remain unchanged for each part are intensive.)

Eg: Pressure, Temperature , Density

Thermodynamic Process

A thermodynamic process is *defined* as a change from one equilibrium state to another.

Quasi-static process

The name quasi-static means nearly static.

A quasi-static process is an infinitely slow process such that the system remains in thermal and mechanical equilibrium with the surroundings throughout.

In a quasi-static process, the pressure and temperature of the environment can differ from those of the system only infinitesimally.

Eg: Processes that are sufficiently slow and do not involve accelerated motion of the piston, large temperature gradient, etc. are reasonably approximation to an ideal quasi-static process.

Some special thermodynamic processes

Type of processes	Feature
Isothermal	Temperature constant
Isobaric	Pressure constant
Isochoric	Volume constant
Adiabatic	No heat flow between the system and the surroundings ($\Delta Q = 0$)

Isothermal process.

A process in which the temperature of the system is kept fixed throughout is called an isothermal process.

For isothermal process **T = constant** .

So internal energy does not change, **$\Delta U=0$**

- Eg: Change of state (Melting, fusion, vaporistion..)
- The expansion of a gas in a metallic cylinder placed in a large reservoir of fixed temperature is an example of an isothermal process.

Equation of state for an isothermal process

For an ideal gas, $PV = \mu RT$

If an ideal gas goes isothermally from its initial state to the final state , its temperature remains constant

$$PV = \text{constant}$$

This is the equation of state for an isothermal process.

Work done by an ideal gas during an isothermal process

Consider an ideal gas undergoes a change in its state isothermally (at temperature T) from (P_1, V_1) to the final state (P_2, V_2).

$$\Delta W = P\,\Delta V$$

In the limit $\Delta V \rightarrow 0$

$$dW = P\,dV$$

$$W = \int_{V_1}^{V_2} P\,dV$$

$$P\,V = \mu R T \text{ (for 1 mole)}$$

$$P = \frac{\mu R T}{V}$$

$$W = \int_{V_1}^{V_2} \frac{\mu R T}{V}\,dV$$

$$W = \mu R T \int_{V_1}^{V_2} \frac{1}{V}\,dV$$

$$W = \mu R T\,[\ln V]_{V_1}^{V_2}$$

$$W = \mu R T\,[\ln V_2 - \ln V_1]$$

$$\mathbf{W = \mu RT \ln \left[\frac{V_2}{V_1}\right]}$$

Isothermal expansion,

For Isothermal expansion, $V_2 > V_1$ and hence $W > 0$ (workdone is positive)

That is, in an isothermal expansion, the gas absorbs heat and work is done by the gas on the environment.

Isothermal compression

In isothermal compression $V_2 < V_1$ and hence $W < 0$ (workdone is negative)

That is, In an isothermal compression, work is done on the gas by the environment and heat is released.

The First Law of Thermodynamics for an isothermal change

For isothermal process $\mathbf{\Delta U = 0}$

First Law of Thermodynamics $\Delta Q = \Delta U + \Delta W$

$$\Delta Q = \Delta W$$

The heat absorbed by the gas is entirely used to do the work on the environment.

Adiabatic process

In an adiabatic process, the system is insulated from the surroundings and heat absorbed or released is zero.

$$\Delta Q = 0$$

Equation of state for an adiabatic process

$$PV^{\gamma} = \text{constant}$$

Or $$T\,V^{\gamma-1} = \text{constant}$$

where γ is the ratio of specific heats (ordinary or molar) at constant pressure and at constant volume. $\gamma = \frac{C_p}{C_v}$

Workdone by an Ideal gas during an Adiabatic Process

Consider an ideal gas undergoes a change in its state adiabatically from (P_1, V_1) to the final state (P_2, V_2).

$$W = \int_{V_1}^{V_2} P\,dV$$

$$PV^{\gamma} = k$$

$$P = \frac{k}{V^{\gamma}}$$

$$P = k\,V^{-\gamma}$$

$$W = k\int_{V_1}^{V_2} V^{-\gamma}\,dV$$

$$W = k\left[\frac{V^{-\gamma+1}}{-\gamma+1}\right]_{V_1}^{V_2}$$

$$W = \frac{k}{1-\gamma}\left[V_2^{-\gamma+1} - V_1^{-\gamma+1}\right]$$

$$W = \frac{1}{1-\gamma}\left[\frac{k}{V_2^{\gamma-1}} - \frac{k}{V_1^{\gamma-1}}\right]$$

$$PV^{\gamma} = k$$

$$P_1V_1^{\gamma} = P_2V_2^{\gamma} = k$$

$$W = \frac{1}{1-\gamma}\left[\frac{P_2V_2^{\gamma}}{V_2^{\gamma-1}} - \frac{P_1V_1^{\gamma}}{V_1^{\gamma-1}}\right]$$

$$\mathbf{W = \frac{1}{1-\gamma}\left[P_2V_2 - P_1V_1\right]}$$

$$PV = \mu RT$$

$$W = \frac{1}{1-\gamma}\left[\mu RT_2 - \mu RT_1\right]$$

$$\mathbf{W = \frac{\mu R}{1-\gamma}\left[T_2 - T_1\right]}$$

Or

$$\mathbf{W = \frac{\mu R}{\gamma-1}\left[T_1 - T_2\right]}$$

Adiabatic expansion

In adiabatic expansion, the work is done by the gas ($W > 0$), we get $T_2 < T_1$ i.e., the temperature of the gas lowers.

Adiabatic compression

In Adiabatic compression, work is done on the gas ($W < 0$), we get $T_2 > T_1$. i.e., the temperature of the gas rises.

First law of thermodynamics for an adiabatic process

$$\Delta Q = \Delta U + \Delta W$$

For adiabatic process, $\Delta Q = 0$

$$\Delta W = -\Delta U$$

i.e., Workdone = - change in internal energy

Isochoric process

In an isochoric process, V is constant.

Workdone in an isochoric process

$$\Delta W = P\,\Delta V$$

For isochoric process, $\Delta V = 0$

$$\Delta W = 0$$

In an isochoric process no work is done on or by the gas.

First law of thermodynamics for an isochoric process

$$\Delta Q = \Delta U + \Delta W$$

For isochoric process, $\Delta W = 0$

$$\Delta Q = \Delta U$$

The heat absorbed by the gas goes entirely to change its internal energy and thereby its temperature.

Isobaric Process

In an isobaric process, P is constant.

Work done by the gas in an Isobaric process

Work done by the gas is

$$\Delta W = P\,\Delta V$$

$$W = P\,(V_2 - V_1)$$

$$W = \mu R\,(T_2 - T_1)$$

First Law of Thermodynamics for an Isobaric Process

Since temperature changes, internal energy also changes. The heat absorbed goes partly to increase internal energy and partly to do work.

$$\Delta Q = \Delta U + \Delta W$$

Cyclic Process

In a cyclic process, the system returns to its initial state.

Since internal energy is a state variable, $\Delta U = 0$ for a cyclic process

First Law of Thermodynamics for Cyclic Process

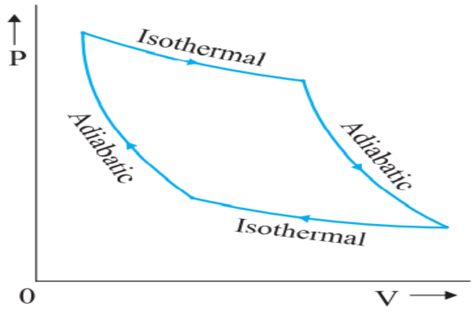
$$\Delta Q = \Delta U + \Delta W$$

$$\Delta U = 0$$

$$\Delta Q = \Delta W$$

The total heat absorbed equals the work done by the system.

P-V curves for isothermal and adiabatic processes of an ideal gas.



Second Law of Thermodynamics

Kelvin-Planck statement

No process is possible whose sole result is the absorption of heat from a reservoir and the complete conversion of the heat into work.

Clausius statement

No process is possible whose sole result is the transfer of heat from a colder object to a hotter object.

The two statements above are completely equivalent

Reversible and Irreversible Processes

Reversible Processes

A thermodynamic process is reversible if the process can be turned back such that both the system and the surroundings return to their original states, with no other change anywhere else in the universe.

Eg: A quasi-static isothermal expansion of an ideal gas in a cylinder fitted with a frictionless movable piston is a reversible process.

A process is reversible only if:

1) It is quasi-static i.e., the system in equilibrium with the surroundings at every stage.
2) There are no dissipative factors such as friction, viscosity, etc.

Irreversible Processes

A thermodynamic process is irreversible if the process cannot be turned back such that both the system and the surroundings return to their original states, with no other change anywhere else in the universe.

The spontaneous processes of nature are irreversible.

- Eg:The free expansion of a gas
- The combustion reaction of a mixture of petrol and air ignited by a spark .
- Cooking gas leaking from a gas cylinder in the kitchen diffuses to the entire room. The diffusion process will not spontaneously reverse and bring the gas back to the cylinder.

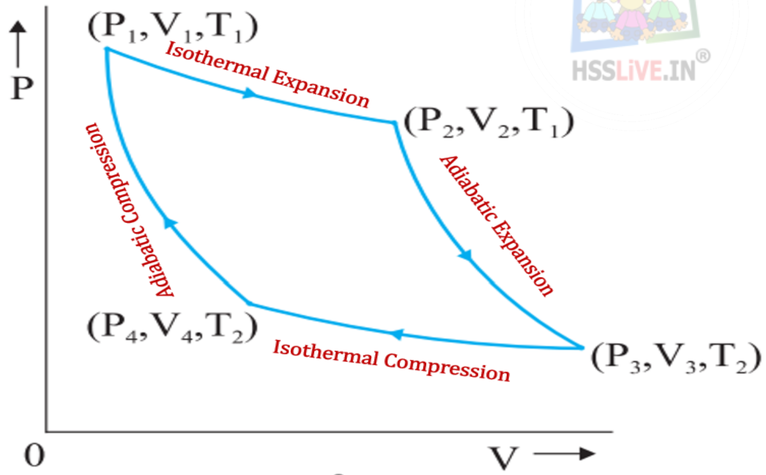
Irreversibility of a process arises due to:

1) Many processes take the system to non-equilibrium states.
2) Most processes involve friction, viscosity and other dissipative effects.

Carnot Engine

Sadi Carnot, a French engineer, developed Carnot engine. Carnot engine is a reversible engine operating between two temperatures T_1 (source) and T_2 (sink). **The working substance of the Carnot engine is an ideal gas.**

Carnot cycle



The four processes involved in carnot cycle are

1.Isothermal Expansion
2. Adiabatic Expansion
3. Isothermal Compression
4. Adiabatic Compression

Workdone in a Carnot cycle

(a)Step 1 →2 Isothermal expansion of the gas from (P_1, V_1, T_1) to (P_2, V_2, T_1).

Since the process is isothermal

The workdone $(W_{1\to2})$ =Heat absorbed (Q_1)

Work done by the gas,

$$W_{1\to2} = Q_1 = \mu RT_1 \ln\left|\frac{V_2}{V_1}\right| \text{------------(1)}$$

(b) Step 2 → 3 Adiabatic expansion of the gas from (P_2,V_2,T_1) to (P_3, V_3, T_2).

Work done by the gas,

$$W_{2\to3} = \frac{\mu R}{\gamma-1}[\,T_1 - T_2\,] \text{ ------------------(2)}$$

(c)Step 3→4 Isothermal compression of the gas from(P_3,V_3,T_2) to (P_4,V_4,T_2).

The workdone $(W_{3\to4})$ =Heat released (Q_2)

Work done on the gas,

$$W_{3\to4} = Q_2 = \mu RT_2 \ln\left|\frac{V_4}{V_3}\right|$$

$$W_{3\to4} = Q_2 = -\mu RT_2 \ln\left|\frac{V_3}{V_4}\right| \text{ -----------(3)}$$

(d) Step 4 → 1 Adiabatic compression of the gas from (P_4,V_4,T_2) to (P_1,V_1, T_1)

Work done on the gas is,

$$W_{4\to1} = \frac{\mu R}{\gamma-1}[T_2 - T_1]$$

$$W_{4\to1} = -\frac{\mu R}{\gamma-1}[T_1 - T_2] \text{ -----------------(4)}$$

Total work done by the gas in one complete cycle is

$$W = W_{1\to2} + W_{2\to3} + W_{3\to4} + W_{4\to1}$$

$$W = \mu RT_1 \ln\left|\frac{V_2}{V_1}\right| + \frac{\mu R}{\gamma-1}[\,T_1 - T_2\,] - \mu RT_2 \ln\left|\frac{V_3}{V_4}\right| - \frac{\mu R}{\gamma-1}[\,T_1 - T_2\,]$$

$$W = \mu RT_1 \ln\left|\frac{V_2}{V_1}\right| - \mu RT_2 \ln\left|\frac{V_3}{V_4}\right| \text{ -------------(5)}$$

Efficiency of Carnot Engine

$$\eta = 1 - \frac{Q_2}{Q_1}$$

$$\eta = 1 - \frac{\mu RT_2 \ln\left|\frac{V_3}{V_4}\right|}{\mu RT_1 \ln\left|\frac{V_2}{V_1}\right|}$$

As the two processes involved are adiabatic ,we get $\frac{V_3}{V_4} = \frac{V_2}{V_1}$

$$\eta = 1 - \frac{T_2}{T_1}$$

$$\eta = \frac{T_1 - T_2}{T_1}$$

Carnot's theorem

(a) working between two given temperatures T_1 and T_2 of the hot and cold reservoirs respectively, no engine can have efficiency more than that of the Carnot engine

(b) the efficiency of the Carnot engine is independent of the nature of the working substance.

Example

Calculate the efficiency of a heat engine working between steam point and ice point. Can you design an engine of 100% efficiency.

Steam point , $T_1 = 100^0C = 100+273 = 373K$

Ice point , $T_2 = 0^0C = 0+273 = 273K$

$$\eta = \frac{T_1 - T_2}{T_1}$$

$$\eta = \frac{373 - 273}{373} = 0.268$$

$$\boldsymbol{\eta = 26.8\%}$$

An ideal engine with $\eta = 1$ or (100%) efficiency is never possible, even if we can eliminate various kinds of losses associated with actual heat engines.

Chapter 12
Kinetic Theory

Introduction

Kinetic theory explains the behaviour of gases based on the idea that the gas consists of rapidly moving atoms or molecules. The kinetic theory was developed in the nineteenth century by Maxwell, Boltzmann and others.

Behaviour of Gases

Ideal gas equation

Gases at low pressures and high temperatures much above that at which they liquefy (or solidify) approximately satisfy a simple relation

$$PV = \mu RT \text{------------}(1)$$

where μ is the number of moles
R is universal gas constant.
$R = 8.314\ J\ mol^{-1}K^{-1}$
T is absolute temperature.

A gas that satisfies this eqn exactly at all pressures and temperatures is defined to be an ideal gas.

But $R = N_A k_B$
N_A is Avogadro number
k_B is Boltzmann constant
$k_B = 1.38 \times 10^{-23}\ J\ K^{-1}$

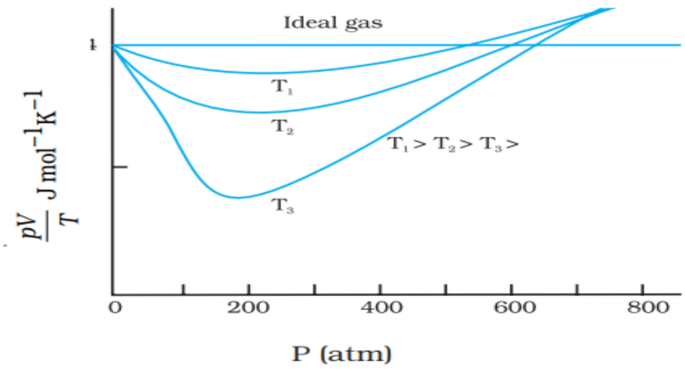
$$PV = \mu N_A k_B T$$

$\mu N_A = N$
N is the number of molecules of gas

The perfect gas equation canalso be written as

$$PV = Nk_BT \text{ ------------}(2)$$

From above eq , $\frac{PV}{T} = \text{constant}$



From graph it is clear that **real gases approach ideal gas behaviour at low pressures and high temperatures.**

At low pressures or high temperatures the molecules are far apart and molecular interactions are negligible. Without interactions the gas behaves like an ideal one.

Boyle's Law

$$PV = \mu RT$$

If we fix μ and T,

$$\mathbf{PV = Constant}$$

$$\mathbf{P \propto \frac{1}{V}}$$

i.e., **for a fixed temperature , pressure of a given mass of gas varies inversely with volume. This is the famous Boyle's law.**

Charles'Law

$$PV = \mu RT$$

If we fix P

$$\frac{\mathbf{V}}{\mathbf{T}} = \mathbf{constant}$$

$$\mathbf{V \propto T}$$

i.e., for a fixed pressure, the volume of a gas is proportional to its absolute temperature T (Charles' law).

Dalton's law of partial pressures.

Consider a mixture of non-interacting ideal gases μ_1 moles of gas 1, μ_2 moles of gas 2, etc

$$PV = (\mu_1 + \mu_2 + \dots)\, RT$$

$$P = \mu_1 \frac{RT}{V} + \mu_2 \frac{RT}{V} + \dots\dots\dots\dots$$

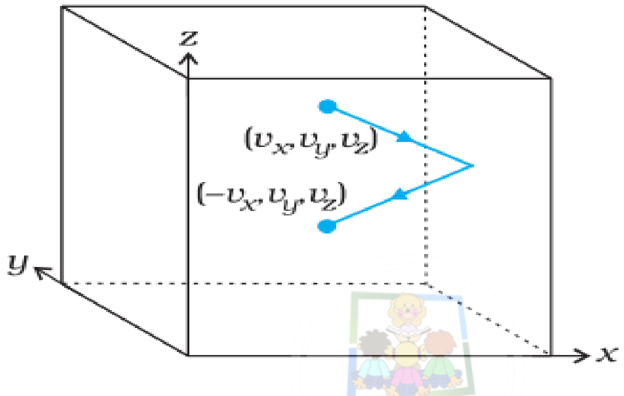
$$\mathbf{P = P_1 + P_2 + \dots\dots\dots\dots}$$

Thus, the total pressure of a mixture of ideal gases is the sum of partial pressures. This is Dalton's law of partial pressures.

Kinetic Theory of an Ideal Gas

- A given amount of gas is a collection of a large number of molecules that are in random motion.
- At ordinary pressure and temperature, the average distance between molecules is very large compared to the size of a molecule (2 Å).
- The interaction between the molecules is negligible.
- The molecules make elastic collisions with each other and also with the walls of the container .
- As the collisions are elastic , total kinetic energy and total momentum are conserved .

Pressure of an Ideal Gas



A gas is enclosed in a cube of side l

Consider a molecule moving in positive x direction makes an elastic collision with the wall of thecontainer.

$$\text{Momentum before collision} = mv_x$$
$$\text{Momentum after collision} = -mv_x$$
$$\text{The change in momentum of the molecule} = -mv_x - mv_x$$
$$= -2\ mv_x$$

By the principle of conservation of momentum

Momentum imparted to wall in the collision $= 2mv_x$

$$\text{Distnace travelled by the molecule in time } \Delta t = v_x\ \Delta t$$
$$\text{Volume covered by the molecule} = Av_x\ \Delta t$$
$$\text{No of molecules in this volume} = n\ Av_x\ \Delta t$$
(n is number density of molecules)

Only half of these molecules move in +x direction

$$\text{No of molecules} = \frac{1}{2} nA\ v_x\ \Delta t$$

The number of molecules with velocity v_x hitting the wall in time Δt

$$= \frac{1}{2} nA\ v_x\ \Delta t$$

The total momentum transferred to the wall

$$Q = (2mv_x)\left(\frac{1}{2} nA\, v_x\, \Delta t\right)$$

$$Q = nmAv_x{}^2\, \Delta t$$

The force on the wall, $F = \frac{Q}{\Delta t}$

$$F = nmAv_x{}^2$$

Pressure, $P = \frac{F}{A}$

$$P = nmv_x{}^2$$

All molecules in a gas do not have the same velocity; so average velocity is to be taken

$$P = nm\overline{v_x^2}$$

$$\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$$
$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$
$$\overline{v^2} = 3\,\overline{v_x^2}$$
$$\overline{v_x^2} = \frac{1}{3}\overline{v^2}$$

but $\overline{v_x^2} = \frac{1}{3}\overline{v^2}$

$$\mathbf{P = \frac{1}{3}\, nm\overline{v^2}}$$

Kinetic Interpretation of Temperature

$$P = \frac{1}{3}\, nm\overline{v^2}$$

$$PV = \frac{1}{3}\, nVm\overline{v^2}$$

$$n = \frac{N}{V}, \quad N = nV$$

$$PV = \frac{1}{3}\, Nm\overline{v^2}$$

where N is the number of molecules in the sample.

$$PV = \frac{2}{3}\left(N\, \frac{1}{2} m\overline{v^2}\right)$$

The quantity in bracket is the average translational kinetic energy of the molecules in the gas.

$$N\, \frac{1}{2} m\overline{v^2} = E$$

$$PV = \frac{2}{3}\, E \text{---------------------}(1)$$

Ideal gas equation, $PV = Nk_BT$ --------------------(2)

From eq(1)and (2) $\quad \frac{2}{3}\, E = Nk_BT$

$$E = \frac{3}{2} Nk_BT$$

$$\mathbf{E/N = \frac{3}{2} k_BT}$$

The average kinetic energy of a molecule is proportional to the absolute temperature of the gas; it is independent of pressure, volume or the nature of the ideal gas.

Note:

This is a fundamental result relating temperature(a macroscopic measurable parameter) to the average kinetic energy(microscopic quantity) of a gas molecule. The two domains are connected by the Boltzmann constant.

Root Mean Square (rms) Speed

$$E/N = \frac{3}{2}k_B T$$

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}k_B T$$

$$\overline{v^2} = \frac{3k_B T}{m}$$

The square root of $\overline{v^2}$ is known as root mean square (rms) speed and is denoted by v_{rms}

$$\mathbf{v_{rms} = \sqrt{\frac{3k_B T}{m}}}$$

Degrees f Freedom

The total number of independent ways in which a system can possess energy is called degree of freedom.

A molecule has one degree of freedom for motion in a line.

Two degrees of freedom for motion in a plane.

Three degrees of freedom for motion in space.

Law of Equipartition of Energy

Law of Equipartition of Energy states that, in equilibrium, the total energy is equally distributed in all possible energy modes, with each mode having an average energy equal to $\mathbf{\frac{1}{2}k_B T}$

- Each translational degree of freedom contributes $\frac{1}{2}k_B T$
- Each rotational degree of freedom contributes $\frac{1}{2}k_B T$
- Each vibrational degree of freedom contributes , $2 x \frac{1}{2}k_B T = k_B T$ as a vibrational mode has both kinetic and potential energy modes.

Specific Heat capacity

Specific Heat capacities of Monoatomic Gases

The molecule of a monatomic gas has only 3 translational degrees of freedom.

$$\text{Average energy of a molecule} = 3 \; x \; \frac{1}{2} k_B T = \frac{3}{2} k_B \, T$$

The total internal energy of a mole of such a gas is,

$$U = \frac{3}{2} k_B \, T \text{ x } N_A$$

$$k_B N_A = R$$

$$U = \frac{3}{2} RT$$

Specific heat capacity at constant volume

$$C_v = \frac{dU}{dt}$$

$$= \frac{d}{dT}\left(\frac{3}{2} RT\right)$$

$$\mathbf{C_V = \frac{3}{2} R}$$

For an ideal gas, $C_P - C_V = R$ (Mayer's relation)

Specific heat capacity at constant pressure,

$$C_P = C_V + R$$

$$= \frac{3}{2} R + R$$

$$\mathbf{C_P = \frac{5}{2} R}$$

The ratio of specific heats

$$\frac{C_P}{C_P} = \gamma = \frac{\frac{5}{2} R}{\frac{3}{2} R}$$

Adiabatic constant , $\boldsymbol{\gamma = \frac{5}{3}}$

Specific heat capacities of Diatomic Gases

Rigid diatomic molecule

A diatomic rigid rotator has , 3 translational and 2 rotational degrees of freedom.i.e.,

$$\text{Average energy of a molecule} = 5 \ \text{x} \frac{1}{2} k_B T$$

$$= \frac{5}{2} k_B \ T$$

The total internal energy of a mole of such a gas is,

$$U = \frac{5}{2} k_B \ T \ \text{x} \ N_A$$

$$k_B N_A = R$$

$$U = \frac{5}{2} RT$$

Specific heat capacity at constant volume

$$C_v = \frac{dU}{dt}$$

$$= \frac{d}{dT}\left(\frac{5}{2} \ RT\right)$$

$$\mathbf{C_V = \frac{5}{2} R}$$

For an ideal gas, $C_P - C_V = R$ (Mayer's relation)

Specific heat capacity at constant pressure,

$$C_P = C_V + R$$

$$= \frac{5}{2} R + R$$

$$\mathbf{C_P = \frac{7}{2} R}$$

The ratio of specific heats

$$\frac{C_P}{C_P} = \gamma = \frac{\frac{7}{2} R}{\frac{5}{2} R}$$

Adiabatic constant , $\boldsymbol{\gamma = \frac{7}{5}}$

Non Rigid Diatomic Molecule

A non rigid diatomic molecule has , 3 translational , 2 rotational and 1 vibrational degrees of freedom.

(Each vibrational degree of freedom contributes ,2x $\frac{1}{2}k_BT = k_BT$)

$$\text{Average energy of a molecule} = \frac{5}{2}k_BT + k_B T$$

$$= \frac{7}{2}k_BT$$

The total internal energy of a mole of such a gas is,

$$U = \frac{7}{2}k_B T \times N_A$$

$$k_BN_A = R$$

$$\mathbf{U = \frac{7}{2}RT}$$

Specific heat capacity at constant volume

$$C_v = \frac{dU}{dt}$$

$$= \frac{d}{dT}\left(\frac{7}{2}RT\right)$$

$$\mathbf{C_V = \frac{7}{2}R}$$

For an ideal gas, $C_P - C_V = R$ (Mayer's relation)

Specific heat capacity at constant pressure,

$$C_P = C_V + R$$

$$= \frac{7}{2}R + R$$

$$\mathbf{C_P = \frac{9}{2}R}$$

The ratio of specific heats

$$\frac{C_P}{C_P} = \gamma = \frac{\frac{9}{2}R}{\frac{7}{2}R}$$

Adiabatic constant , $\boldsymbol{\gamma = \frac{9}{7}}$

Polyatomic Gases

A polyatomic molecule has 3 translational, 3 rotational degrees of freedom and a certain number (f) of vibrational modes.

$$\text{Average energy of a molecule} = \frac{3}{2}k_BT + \frac{3}{2}k_BT + f\,k_BT$$

The total internal energy of a mole of such a gas is,

$$U = \left(\frac{3}{2}k_BT + \frac{3}{2}k_BT + f\,k_BT\right)N_A$$

$$U = (3 + f)k_BTN_A$$

$$k_BN_A = R$$

$$U = (3 + f)RT$$

Specific heat capacity at constant volume

$$C_v = \frac{dU}{dt}$$

$$\mathbf{C_V = (3 + f)R}$$

Specific heat capacity at constant pressure,

$$C_P = C_V + R$$

$$= (3 + f)R + R$$

$$\mathbf{C_P = (4 + f)R}$$

The ratio of specific heats

$$\gamma = \frac{C_P}{C_P} = \frac{(4+f)R}{(3+f)R}$$

Adiabatic constant , $\boldsymbol{\gamma = \frac{(4+f)}{(3+f)}}$

Specific Heat Capacity of Solids

Consider a solid of N atoms, each vibrating about its mean position.

A vibration in one dimension has average energy $= 2x\ \frac{1}{2}k_BT$

$$= k_BT$$

In three dimensions, the average energy $= 3k_BT$

The total internal energy of a mole of solid is,

$$U = 3k_BT \times N_A$$

$$k_BN_A = R$$

$$U = 3RT$$

Specific heat capacity $C = \frac{dU}{dt}$

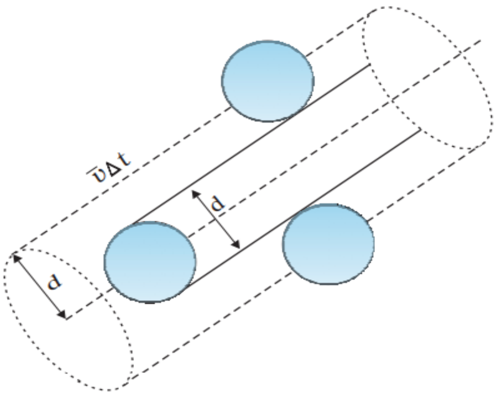
$$= \frac{d}{dT}(3\,RT)$$

$$\mathbf{C = 3R}$$

Mean Free Path

The mean free path l is the average distance covered by a molecule between two successive collisions.

$$l = \frac{1}{\sqrt{2} n \pi d^2}$$



Consider the molecule of a gas as spheres of diameter d. Focus a single molecule.

The volume swept by a molecule in time Δt in which any molecule will collide with it $=$ Area x distancd covered

$$= \pi d^2 \times \langle v \rangle \Delta t$$

The number of collisions suffered by the molecule in time Δt.

$$= n\pi\langle v\rangle d^2 \Delta t$$

where n is the number of molecules per unit volume,

$$\text{Rate of collision} = n\pi\langle v\rangle d^2$$

Time between two successive collisions is on the average,

$$\tau = \frac{1}{n\pi\langle v\rangle d^2}$$

The average distance between two successive collisions, called the mean free path l, is

$$l = \langle v\rangle \tau$$

$$l = \langle v\rangle \frac{1}{n\pi\langle v\rangle d^2}$$

$$l = \frac{1}{n\pi d^2}$$

As all molecules are moving , average relative velocity of the moleculesis to be considered. So the exact equation is

$$l = \frac{1}{\sqrt{2} n \pi d^2}$$

Chapter 13
Oscillations

Non Periodic Motion

The motion which is non-repetitive .
e.g. rectilinear motion , motion of a projectile.

Periodic Motion

A motion that repeats itself at regular intervals of time is called periodic motion.
e.g. uniform circular motion , orbital motion of planets in the solar system.

Oscillatory Motion

Periodic to and fro motion is called oscillatory motion.
e.g. motion of a cradle , motion of a swing, motion of the pendulum of a wall clock.

Every oscillatory motion is periodic, but every periodic motion need not be oscillatory.

Oscillations and Vibration

There is no significant difference between oscillations and vibrations.

- **When the frequency is small, we call it oscillation.**

  e.g.The oscillation of a branch of a tree

- **When the frequency is high, we call it vibration.**

  e.g. The vibration of a string of a musical instrument.

Period and frequency

Period (T)

The period T is the time required for one complete oscillation, or cycle.

Its SI unit is second.

Frequency

The frequency ν of periodic or oscillatory motion is the number of oscillations per unit time.

It is the reciprocal of period .

$$\nu = \frac{1}{T}$$

The SI unit of ν is hertz (Hz).
(In honor of the discoverer of radio waves, Heinrich Rudolph Hertz)

$$1\text{Hz} = 1\text{oscillation per second} = 1\text{s}^{-1}$$

Example

On an average a human heart is found to beat 75 times in a minute. Calculate its frequency and period.

$$\text{The beat frequency of heart}, \nu = \frac{75}{1\text{min}}$$

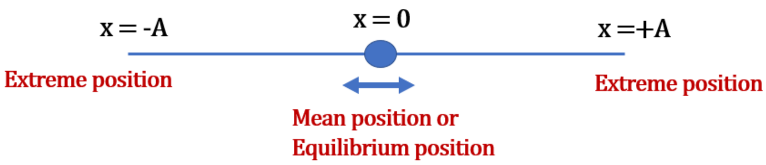
$$= \frac{75}{60\text{ s}}$$

$$= 1.25\text{ s}^{-1} = 1.25\text{ Hz}$$

$$\text{The time period}, T = \frac{1}{1.25}$$

$$T = 0.8\text{ s}$$

Displacement



The distance from mean position is called displacement (x)

At mean position displacement x= 0 and at extreme position x= $\pm A$

A is called amplitude of oscillation.

Amplitude

The maximum displacement from the mean poition is called amplitude (A) of oscillation.

Mathematical Expression for Displacement

The displacement can be represented by a mathematical function of time. It can be a sine function, cosine function or a linear combination of sine and cosine functions.

$$f(t) = A\cos\omega t \quad \text{or}$$

$$f(t) = A\sin\omega t.$$

$$f(t) = A\sin\omega t + B\cos\omega t$$

Where A = Amplitude

ω=angular frequency

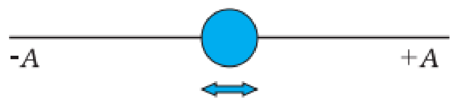
$$\omega = \frac{2\pi}{T} \text{ or } \omega = 2\pi\nu$$

Simple Harmonic Motion

Simple harmonic motion is the simplest form of oscillatory motion.

A particle is said to be in simple harmonic motion ,if the force acting on the particle is proportional to its displacement and is directed towards the mean position.

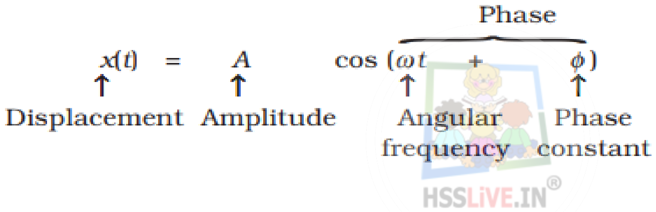
Mathematical expression for an SHM



Consider a particle vibrating back and forth about the origin of x-axis, between the limits +A and –A.

If the motion is simple harmonic ,its position can be represented as a function of time.

$$x(t) = A\cos(\omega t + \phi)$$



Phase

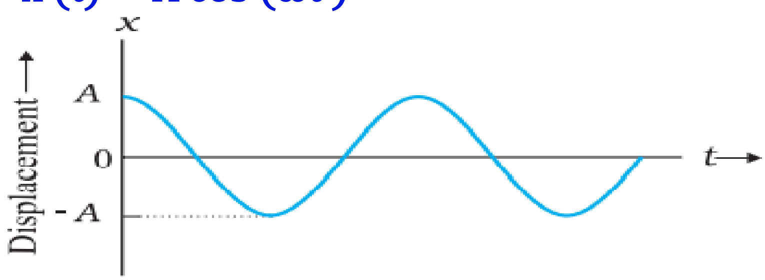
The time varying quantity, **$(\omega t + \phi)$**, is called the phase of the motion. It describes the state of motion at a given time.

Phase Constant

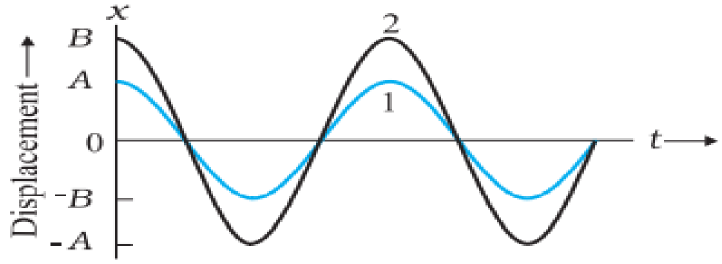
The constant ϕ is called the phase constant (or phase angle). The value of ϕ depends on the displacement and velocity of the particle at $t = 0$. The phase constant signifies the initial conditions.

A plot of displacement as a function of time for $\phi = 0$.

$$x(t) = A\cos(\omega t)$$



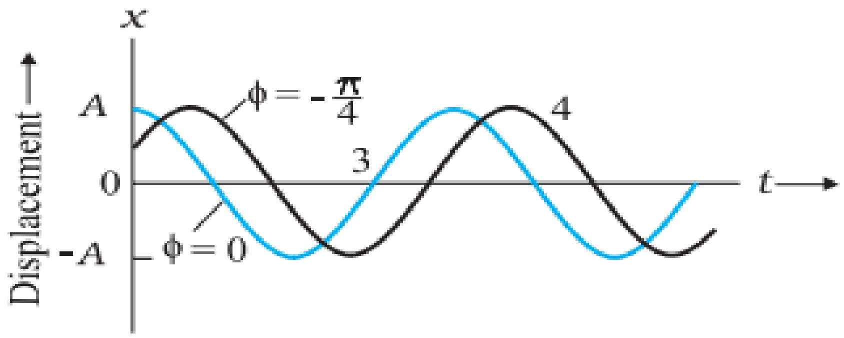
The curves 1 and 2 are for two different amplitudes A and B.



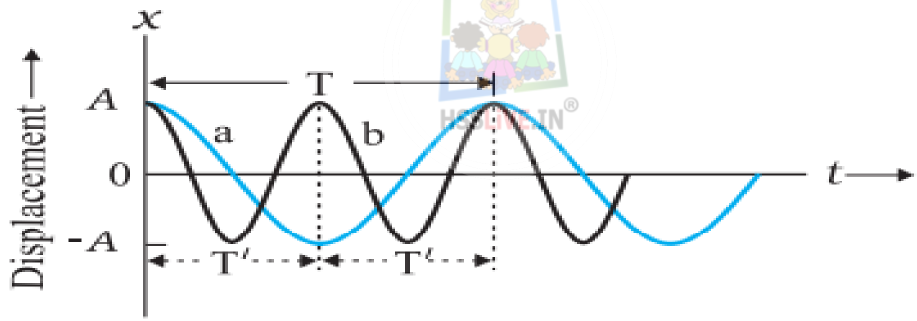
The curve 3 , for $\phi = 0$, $x(t) = A\cos(\omega t)$

The curve 4 , for $\phi = -\pi/4$, $x(t) = A\cos(\omega t - \pi/4)$

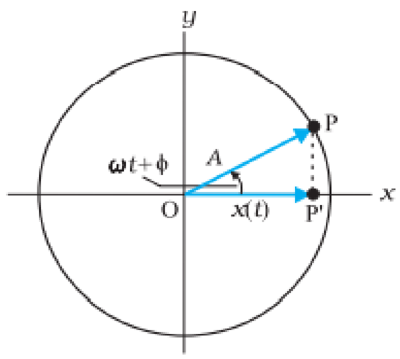
The amplitude A is same for both the plots



Plots of for $\phi = 0$ for two different periods.



Simple Harmonic Motion and Uniform Circular Motion



Consider a particle P in uniform circular motion.

The projection of particle along a diameter of the circle is x(t) .

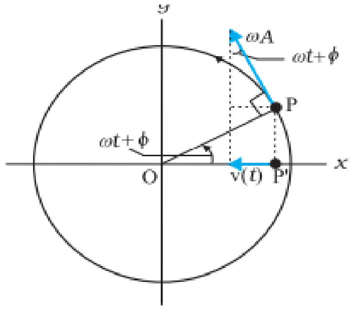
From figure, $\cos(\omega t + \phi) = \frac{x(t)}{A}$

$$x(t) = A\cos(\omega t + \phi) \text{ ----------------(1)}$$

This equation represents a Simple Harmonic Motion.

i. e, the projection of uniform circular motion on a diameter of the circle is in Simple Harmonic Motion.

Velocity in Simple Harmonic Motion



The magnitude of the velocity of the particle is ωA ; its projection on the x-axis gives the velocity of SHM

Displacement in SHM is , $x = A\cos(\omega t + \phi)$

Velocity in SHM can be obtained by differentiating x.

$$v = \frac{d}{dt}x$$

$$= \frac{d}{dt}[A\cos(\omega t + \phi)]$$

$$= A\, x\, -\sin(\omega t + \phi)\, x\, \omega$$

$$v = -\,\omega A\sin(\omega t + \phi)$$

$$\sin^2(\omega t + \phi) = 1 - \cos^2(\omega t + \phi)$$

$$\sin(\omega t + \phi) = \sqrt{1 - \cos^2(\omega t + \phi)}$$

$$v = -\omega A\sqrt{1 - \cos^2(\omega t + \phi)}$$

$$v = -\omega\sqrt{A^2 - A^2\cos^2(\omega t + \phi)}$$

$(x = A\cos(\omega t + \phi))$

$$v = -\,\omega\sqrt{A^2 - x^2} \text{ ------------(2)}$$

Case 1 -At Mean position

$$x = 0$$

$$v = \omega\sqrt{A^2 - 0}$$

$$v = \omega A$$

Velocity is maximum At Mean position

Case 2 -At extreme position

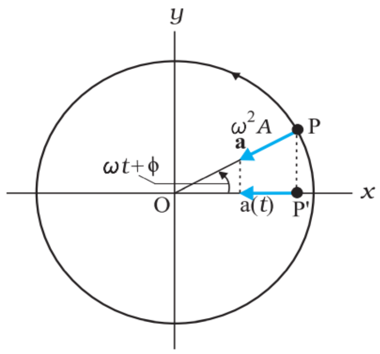
$$x = A$$

$$v = \omega\sqrt{A^2 - A^2}$$

$$v = 0.$$

The velocity is minimum at extreme positions.

Acceleration in SHM



The radial acceleration of Particle is is $\omega^2 A$. Its projection on the x-axis gives the acceleration in SHM.

Acceleration in SHM can be obtained by differentiating velocity in SHM.

$$a=\frac{dv}{dt}$$

$$a=\frac{d}{dt}-\omega A\sin(\omega t+\phi)$$

$$a=-\omega A\cos(\omega t+\phi)\ x\ \omega$$

$$a=-\omega^2 A\cos(\omega t+\phi)$$

$$x=A\cos(\omega t+\phi)$$

$$\mathbf{a=-\omega^2 x \text{-------------}(3)}$$

In SHM, the acceleration is proportional to the displacement and is always directed towards the mean position.

Case 1 -At Mean position

$$x=0$$
$$a=-\omega^2 x$$
$$\mathbf{a=0}$$

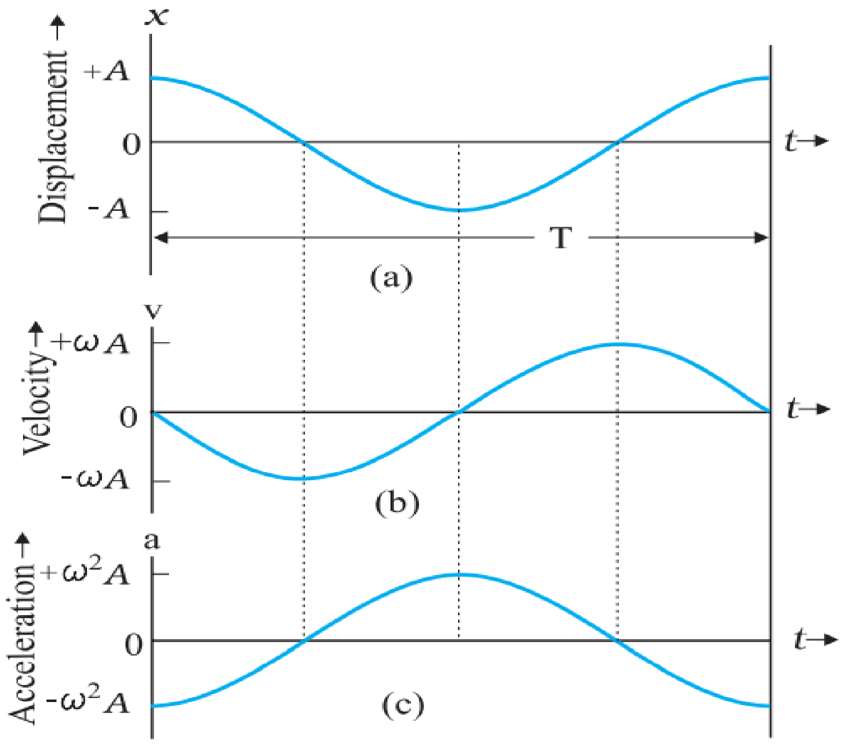
Magnitude of acceleration is minimum at mean position.

Case 2 -At extreme position

$$x=A$$
$$a=-\omega^2 x$$
$$\mathbf{a=-\omega^2 A}$$

The acceleration is maximum at extreme positions.

The variation of particle displacement, velocity and acceleration in a simple harmonic motion



Force Law for Simple Harmonic Motion

$$F = ma$$

$$a = -\omega^2 x$$

$$F = -m\omega^2 x$$

$$\mathbf{F = -kx} \text{ -------------(4)}$$

Where $k = m\omega^2$; $\boldsymbol{\omega^2 = \frac{k}{m}}$

$$\boldsymbol{\omega = \sqrt{\frac{k}{m}}}$$

The force in SHM is proportional to the displacement and its direction is opposite to the direction of displacement. Therefore, it is a restoring force.

Note:

- The centripetal force for uniform circular motion is constant in magnitude, but the restoring force for SHM is time dependent.
- Since the force F is proportional to x such a system is also referred to as a linear harmonic oscillator.

Energy in Simple Harmonic Motion

A particle executing simple harmonic motion has kinetic and potential energies, both varying between the limits, zero and maximum.

Kinetic Energy in Simple Harmonic Motion

$$K = \frac{1}{2}mv^2$$

$$K = \frac{1}{2}mv^2$$

$$v = -\omega\sqrt{A^2 - x^2}$$

$$v^2 = \omega^2(A^2 - x^2)$$

$$\mathbf{K = \frac{1}{2}m\omega^2(A^2 - x^2)} \text{ ----------------(5)}$$

Case 1 -At mean position

$$x=0$$

$$K = \frac{1}{2}m\omega^2(A^2 - 0)$$

$$\mathbf{K = \frac{1}{2}m\omega^2 A^2}$$

KE is maximum At Mean position

Case 2 -At extreme position

$$x=A$$

$$K = \frac{1}{2}m\omega^2(A^2 - A^2)$$

$$\mathbf{K= 0.}$$

KE is minimum At extreme positions.

Thus the kinetic energy of a particle executing simple harmonic motion is periodic, with period T/2.

Potential Energy in Simple Harmonic Motion

$$U = \frac{1}{2}kx^2$$

$$k = m\,\omega^2$$

$$\mathbf{U = \frac{1}{2}m\omega^2 x^2} \text{ ----------------(6)}$$

Case 1 -At Mean position

$$x=0$$

$$U = \frac{1}{2}m\omega^2 x^2$$

$$\mathbf{U = 0}$$

PE is minimum At Mean position

Case 2 -At Extreme position

$$x=A$$

$$U=\frac{1}{2}m\omega^2A^2$$

PE is maximum At extreme positions.

Thus the potential energy of a particle executing simple harmonic motion is also periodic, with period T/2.

The Total Energy in SHM

$$E=U+K$$

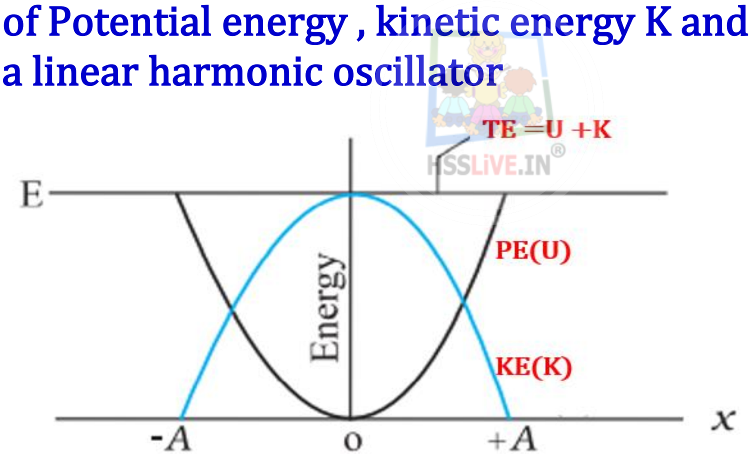
$$E=\frac{1}{2}m\omega^2x^2+\frac{1}{2}m\omega^2(A^2-x^2)$$

$$E=\frac{1}{2}m\omega^2x^2+\frac{1}{2}m\omega^2A^2-\frac{1}{2}m\omega^2x^2$$

$$E=\frac{1}{2}m\omega^2A^2 \text{ ---------------(7)}$$

The total mechanical energy of a harmonic oscillator is a constant or independent of time.

Variation of Potential energy , kinetic energy K and the total energy E with time t for a linear harmonic oscillator



At what position the KE of a simple harmonic oscillator becomes equal to its potential energy?

$$KE=PE$$

$$\frac{1}{2}m\omega^2(A^2-x^2)=\frac{1}{2}m\omega^2x^2$$

$$A^2-x^2=x^2$$

$$A^2=2x^2$$

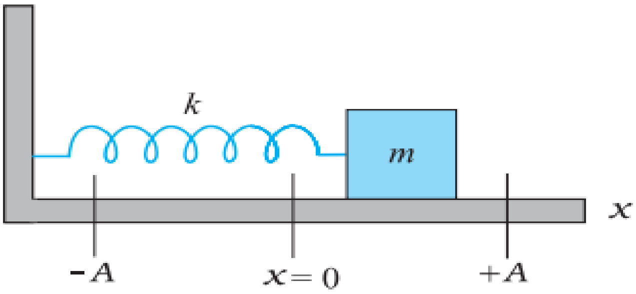
$$x^2=\frac{A^2}{2}$$

$$x=\frac{A}{\sqrt{2}}$$

Some Systems Executing Simple Harmonic Motion

There are no physical examples of absolutely pure simple harmonic motion. In practice we come across systems that execute simple harmonic motion approximately under certain conditions.

Oscillations due to a Spring



The small oscillations of a block of mass m fixed to a spring, is fixed to a rigid wall is an example of SHM.

The restoring force F acting on the block is, F (x) = –k x

k, is called the spring constant.

A stiff spring has large k and a soft spring has small k.

Equation is same as the eqn for force in SHM and therefore the spring executes a simple harmonic motion.

Period of Oscillations of a Spring

Restoring force, $F = -k\,x$

Where $k = m\omega^2$

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\text{Period, } T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$\mathbf{T = 2\pi\sqrt{\frac{m}{k}}}$$

Example

A 5 kg collar is attached to a spring of spring constant 500 N m^{-1}. It slides without friction over a horizontal rod. The collar is displaced from its equilibrium position by 10.0 cm and released. Calculate
(a) the period of oscillation,
(b) the maximum speed and
(c) maximum acceleration of the collar.

(a) The period of oscillation as given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{5}{500}}$$

$$T = 2x\ 3.14\ x\frac{1}{10}$$

$$= 0.63\ s$$

(b) The velocity of the collar executing SHM is

$$v = -\ \omega\sqrt{A^2 - x^2}$$

Maximum speed, $v = A\omega$ (at mean position ,x=0)

$$\omega = \sqrt{\frac{k}{m}}$$

$$v = A\sqrt{\frac{k}{m}}$$

$$A=10cm = 0.1m$$

$$v = 0.1\ x\sqrt{\frac{500}{5}}$$

$$v= 0.1\ x10 =1\ m/s$$

(c) Acceleration in SHM

$$a=-\ \omega^2x$$

Maximum acceleration, $a= \omega^2A$ (at extreme position)

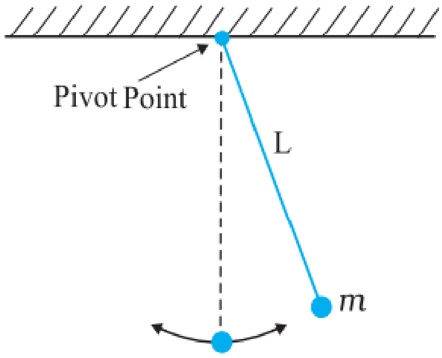
$$\omega^2 = \frac{k}{m}$$

$$a= \frac{k}{m}A$$

$$a= \frac{500}{5}x\ 0.1$$

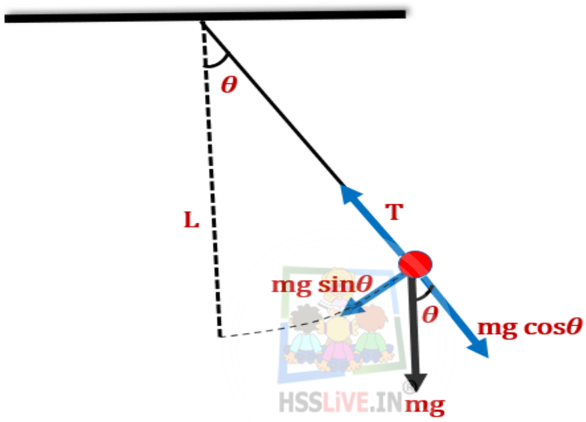
$$a=10m/s^2$$

The Simple Pendulum



A simple pendulum consists of a particle of mass m (bob) suspended from one end of an unstretchable, massless string of length L fixed at the other end.

Period of Oscillations of a Simple Pendulum



The radial component, **mg cosθ** is cancelled by the tension,T.

The tangential component, mg sinθ produces a restoring torque,

$$\tau = -L\,(mg \sin\theta\,) \text{ --------------(1)}$$

(Where the negative sign indicates that the torque acts to reduce θ.)

L= length of simple pendulum.

For rotational motion we have,

$$\tau = I\,\alpha \text{ -----------------(2)}$$

α is angular acceleration.

From eqn (1) and (2)

$$I\,\alpha = -L\,mg \sin\theta$$

$$\alpha = \frac{-mgL}{I}\sin\theta \qquad \text{(since } \theta \text{ is very small, } \sin\theta \approx \theta)$$

$$\alpha = \frac{-mgL}{I}\,\theta \text{ ---------------(3)}$$

Acceleration of SHM , $a = -\,\omega^2 x$ ------------------(4)

Comparing eqns (3) and (4)

$$\omega^2 = \frac{mgL}{I}$$

$$I = mL^2$$

$$\omega^2 = \frac{mgL}{mL^2}$$

$$\omega^2 = \frac{g}{L}$$

$$\omega = \sqrt{\frac{g}{L}}$$

Period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{L}}}$

$$\mathbf{T = 2\pi\sqrt{\frac{L}{g}}}$$

Example

What is the length of a simple pendulum, which ticks seconds (seconds pendulum) ?

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$T^2 = 4\pi^2 \frac{l}{g}$$

$$L = \frac{T^2 g}{4\pi^2}$$

For seconds pendulum ,$T = 2s$

$$L = \frac{2^2 \text{ x } 9.8}{4 \text{ x } 3.14^2} = 0.994 \approx 1m$$

Chapter 14
Waves

Waves

The patterns, which move without the actual physical transfer or flow of matter as a whole, are called waves.

The waves we come across are mainly of three types:

(a) Mechanical waves,
(b) Electromagnetic waves and
(c) Matter waves.

Mechanical waves

Mechanical waves are governed by Newton's laws, and require a material medium for their propagation., such as water, air, rock, etc.
E.g, water waves, sound waves, seismic waves, etc.

Electromagnetic waves

The electromagnetic waves do not require any medium for their propagation.
All electromagnetic waves travel through vacuum at the same speed of light c, 3x 10^8 ms^{-1}
E.g, visible light, ultraviolet light, radio waves, microwaves, x-rays etc.

Matter waves

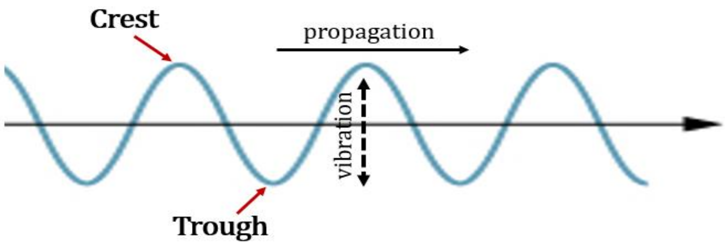
Matter waves are associated with moving electrons, protons, neutrons and other fundamental particles, and even atoms and molecules. These are the constituents of matter and hence such wave are called matter waves. Matter waves associated with electrons are employed in electron microscopes.

Transverse and Longitudinal Waves

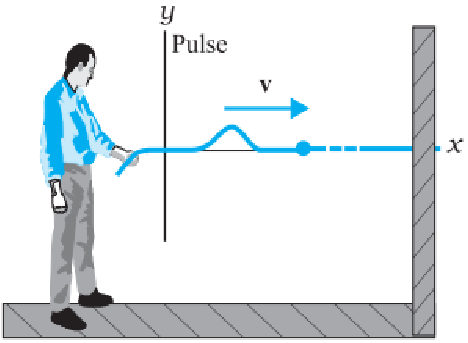
Mechanical waves can be transverse or longitudinal depending on the relationship between the directions of vibrations of particles in the medium and that of the propagation of wave.

Transverse waves

In transverse waves, the constituents of the medium oscillate perpendicular to the direction of wave propagation.



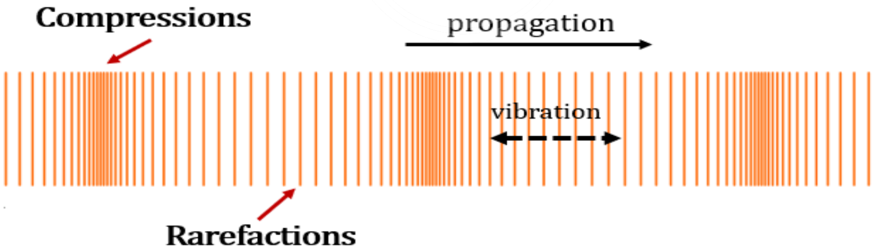
- They travel in the form of crests and troughs.
- Transverse waves can be propagated only in solids and strings, and not in fluids.
- E.g, Waves on a stretched string,



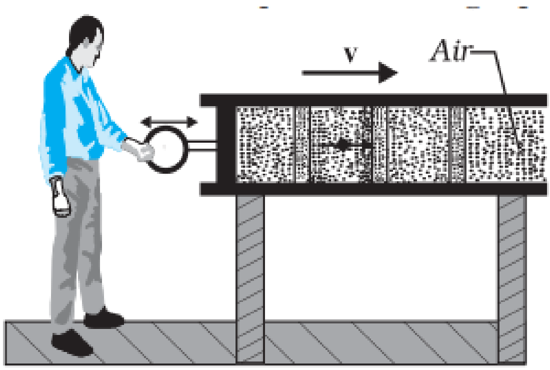
A single pulse is sent along a stretched string. As each element of the string move perpendicular to the direction in which the wave travels, the wave is a transverse wave.

Longitudinal waves

In longitudinal waves the constituents of the medium oscillate along the direction of wave propagation.



- They travel in the form of compressions and rarefactions.
- Longitudinal waves can propagate in all elastic media,i.e,solids,liquids nd gases.
- E.g, sound waves, vibrations in a spring.



A sound wave is set up in an air filled pipe by moving a piston back and forth. As the oscillations of an element of air are parallel to the direction in which the wave travels, the wave is a longitudinal wave.

Capillary waves and Gravity waves

The waves on the surface of water are of two kinds: capillary waves and gravity waves.

Capillary waves

Capillary waves are ripples of fairly short wavelength, not more than a few centimetres. The restoring force that produces them is the surface tension of water.

Gravity waves

Gravity waves have wavelengths typically ranging from several metres to several hundred metres. The restoring force that produces these waves is the pull of gravity, which tends to keep the water surface at its lowest level.

Travelling or Progressive Wave

A wave, transverse or longitudinal, is said to be travelling or progressive if it travels from one point of the medium to another.

Displacement Relation in a Progressive Wave along a String (transverse wave)

A progressive wave travelling along the positive direction of the x-axis can be represented as

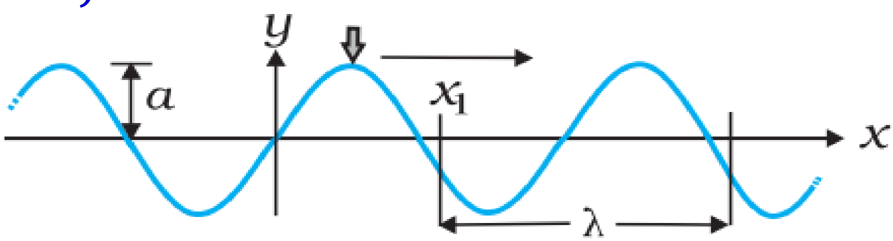
$$y(x, t) = a \sin(kx - \omega t + \phi)$$

A progressive wave travelling along the negative direction of the x-axis can be represented as

$$y(x, t) = a \sin(kx + \omega t + \phi)$$

$$\overbrace{y(x,t)}^{\text{Displacement}} = \overbrace{a}^{\text{Amplitude}} \sin(\overbrace{\underset{\substack{\uparrow \\ \text{Angular} \\ \text{Wave} \\ \text{Number}}}{kx} - \underset{\substack{\uparrow \\ \text{Angular} \\ \text{Frequency}}}{\omega t} + \underset{\substack{\uparrow \\ \text{Initial} \\ \text{Phase} \\ \text{Angle}}}{\phi}}^{\text{Phase}})$$

Graphical variation of displacement wih time for a progressive wave (Transverse wave)



Crest

A point of maximum positive displacement in a wave, is called crest.

Trough

A point of maximum negative displacement is called trough.

Amplitude

The amplitude a of a wave is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them.

Since ‘**a**’ is a magnitude, it is a positive quantity, even if the displacement is negative.

Phase

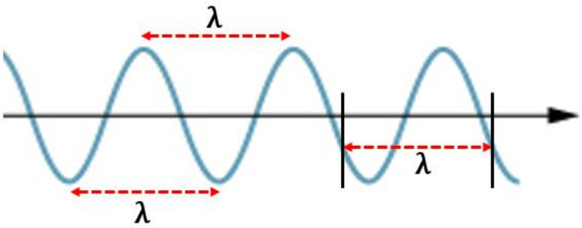
The phase of the wave is the argument $(kx - \omega t + \phi)$ of the oscillatory term sine . It describes the state of motion as the wave sweeps through a string element at a particular position x. It changes linearly with time t.

Phase Constant

The constant ϕ is called the initial phase angle. The value of ϕ is determined by the initial displacement(at, t=0)and velocity of the element (at, x = 0).

Wavelength and Angular Wave Number

Wavelength



The wavelength λ of a wave is the minimum distance between two consecutive troughs or crests or two consecutive points in the same phase of wave motion.

Propagation Constant or Angular Wave Number

Propagation constant or Angular Wave Number is defined as

$$k = \frac{2\pi}{\lambda}$$

Its SI unit is radian per metre or rad m^{-1}

Period, Angular Frequency and Frequency

Period

The period of oscillation T of a wave is defined as the time taken by any element to complete one oscillation.

Angular Frequency

Angular Frequency of a wave is given by

$$\omega = \frac{2\pi}{T}$$

Its SI unit is rad s^{-1}

From this equation, $T = \frac{2\pi}{\omega}$

Frequency

It is the number of oscillations per unit time made by an element as the wave passes through it.

Frequency is the reciprocal of period.

$$\nu = \frac{1}{T}$$

$$\nu = \frac{\omega}{2\pi}$$

It is usually measured in hertz.

Example

A wave travelling along a string is described by,
$y(x, t) = 0.005 \sin(80.0\,x - 3.0\,t)$, in which the numerical constants are in SI units (0.005 m, 80.0 rad m^{-1}, and 3.0 rad s^{-1}). Calculate

(a) the amplitude,
(b) the wavelength,
(c) the period and frequency of the wave.
(d) Calculate the displacement y of the wave at a distance $x = 30.0$ cm and time $t = 20$ s ?

Answer

$$y(x, t) = 0.005 \sin(80.0\,x - 3.0\,t)$$

The general expression for a travelling wave is

$$y(x,t) = a \sin(kx - \omega t + \phi)$$

Comparing these equations

(a) Amplitude , **a=0.005m**

(b) k=80 rad m^{-1}

but , $k = \frac{2\pi}{\lambda}$

$$\frac{2\pi}{\lambda} = 80$$

$$\lambda = \frac{2\pi}{80} = 0.0785 \text{ m}$$

$$\boldsymbol{\lambda = 7.85\ cm}$$

(c) ω=3

$$\frac{2\pi}{T} = 3$$

$$T = \frac{2\pi}{3} = \mathbf{2.09\ s}$$

Frequency, $\nu = 1/T = 1/2.09$

$$\boldsymbol{\nu = 0.48\ \text{Hz}}$$

(d) $y(x, t) = 0.005 \sin(80.0\,x - 3.0\,t)$

$x = 30.0$ cm $= 0.3$m

$t = 20$ s

$$y(x, t) = 0.005 \sin(80.0 \times 0.3 - 3.0 \times 20)$$

$$= (0.005 \text{ m}) \sin(-36)$$

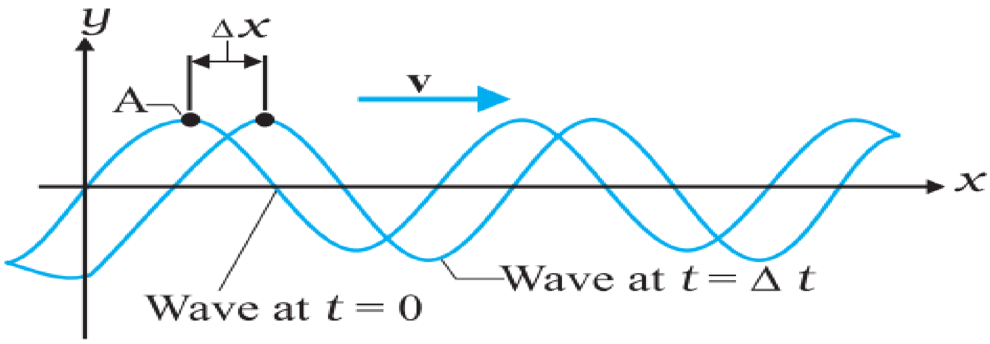
$$= (0.005 \text{ m}) \sin(-36 + 12\pi)$$

12 π is added ,so tht (−36 + 12 π) becomes positive

$$= (0.005 \text{ m}) \sin(1.699)$$

$$= (0.005 \text{ m}) \sin(97^0) = 5 \text{ mm}$$

The Speed of a Travelling Wave



Consider a wave propagating in positive x direction with initial phase $\phi = 0$

$$y(x, t) = a \sin(kx - \omega t)$$

As the wave moves, each point of the waveform (say A) retains its displacement y. This is possible only when the argument $(kx - \omega t)$ is constant.

$$(kx - \omega t) = \text{constant}$$

$$\frac{d}{dt}(kx - \omega t) = 0$$

$$k\frac{dx}{dt} - \omega\frac{dt}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k}$$

$$\mathbf{v = \frac{\omega}{k}}$$

$$\omega = 2\pi\nu \ , \quad k = \frac{2\pi}{\lambda}$$

$$v = \frac{2\pi\nu}{\frac{2\pi}{\lambda}}$$

$$\mathbf{v = \nu\lambda}$$

This is a general relation valid for all progressive waves.

The speed of a wave is related to its wavelength and frequency , but it is determined by the properties of the medium.

Speed of a Transverse Wave on Stretched String

The speed of transverse waves on a string is determined by two factors,

(i) the linear mass density or mass per unit length, μ, and

(ii) the tension T

$$\mathbf{v = \sqrt{\frac{T}{\mu}}}$$

The speed of a wave along a stretched ideal string does not depend on the frequency of the wave.

Example

A steel wire 0.72 m long has a mass of 5.0×10^{-3} kg. If the wire is under a tension of 60 N, what is the speed of transverse waves on the wire ?

$$v=\sqrt{\frac{T}{\mu}}$$

$$\mu = \frac{M}{l}$$

$$= \frac{5.0 \times 10^{-3}}{0.72}$$

$$= 6.9 \times 10^{-3} \text{ kg } m^{-1}$$

$$T = 60 \text{ N}$$

$$v=\sqrt{\frac{60}{6.9 \times 10^{-3}}} = 93 \text{ m } s^{-1}$$

Speed of a Longitudinal Wave(Speed of Sound)

The longitudinal waves in a medium travel in the form of compressions and rarefactions or changes in density, ρ.

- **The speed of propagation of a longitudinal wave in a fluid**

$$v=\sqrt{\frac{B}{\rho}}$$

B= the bulk modulus of medium

ρ = the density of the medium

- **The speed of a longitudinal wave in a solid bar**

$$v=\sqrt{\frac{Y}{\rho}}$$

Y =Young's modulus

ρ=density of the medium,

- **The speed of a longitudinal wave in an ideal gas**

Case1 -Newtons Formula

Newton assumed that, the pressure variations in a medium during propagation of sound are isothermal.

$$v=\sqrt{\frac{B}{\rho}}$$

For isothermal process

$$PV = \text{constant}$$

$$V\Delta P + P\Delta V = 0$$

$$V\Delta P = -P\Delta V$$

$$-\frac{V\Delta P}{\Delta V} = P$$

$$B = P$$

$$v=\sqrt{\frac{P}{\rho}}$$

This relation was first given by Newton and is known as **Newton's formula.**

Case 2- Laplace correction to Newton's formula.

Laplace that the pressure variations in the propagation of sound waves are so fast that there is little time for the heat flow to maintain constant temperature. These variations, therefore, are adiabatic and not isothermal.

$$v=\sqrt{\frac{B}{\rho}}$$

For adiabatic processes

$PV^{\gamma} = \text{constant}$

$\Delta\, PV^{\gamma} = 0$

$P\gamma V^{\gamma-1}\Delta V + V^{\gamma}\,\Delta P = 0$

$\gamma P\, V^{\gamma-1}\Delta V = -V^{\gamma}\,\Delta P$

$\gamma\, P = -\frac{V^{\gamma}\Delta P}{V^{\gamma-1}\Delta V}$

$\gamma\, P = -\frac{\Delta P}{V^{-1}\Delta V}$

$\gamma\, P = -\frac{V\Delta P}{\Delta V} = B$

$B = \gamma\, P$

$$v=\sqrt{\frac{\gamma P}{\rho}}$$

This modification of Newton's formula is referred to as the **Laplace correction.**

$\gamma = \frac{C_P}{C_V}$, For air $\gamma = \frac{7}{5}$.

The speed of sound in air at STP = 331.3 m s^{-1}

The Principle of Superposition of Waves

When two or more waves overlap, the resultant displacement is the algebraic sum of the displacements due to each wave.

let y_1(x, t) and y_2 (x, t) be the displacements individual waves, then resultant displacement y (x, t) is,

$$y(x, t) = y_1(x, t) + y_2(x, t)$$

Consider two wave travelling in positive x direction having same amplitude, same angular frequency and wavenumber and therefor same wavelength and speed. The waves differ only in their initial phase ϕ

$$y_1(x, t) = a \sin(kx - \omega t)$$
$$y_2(x, t) = a \sin(kx - \omega t + \phi)$$

Now, applying the superposition principle, the resultant displacement is

$$y(x, t) = a \sin(kx - \omega t) + a \sin(kx - \omega t + \phi)$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Applying this relation the resultant displacement is,

$$y(x, t) = \left[2a \cos\frac{\phi}{2}\right] \sin\left(kx - \omega t + \frac{\phi}{2}\right) \text{----------(1)}$$

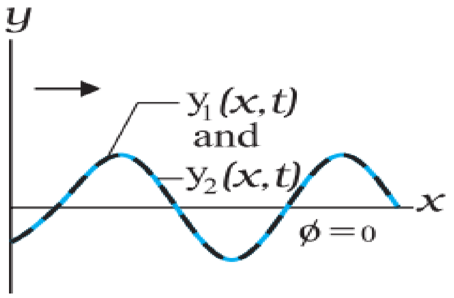
The resultant wave is also a sinusoidal wave, travelling in the positive direction of x-axis.

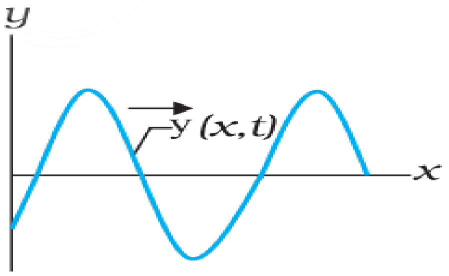
Initial phase of resultant wave = $\frac{\phi}{2}$

Amplitude of resultant wave is

$$A = 2a \cos\frac{\phi}{2} \text{------------(2)}$$

If $\phi = 0$, i.e. ,the two waves are in phase,



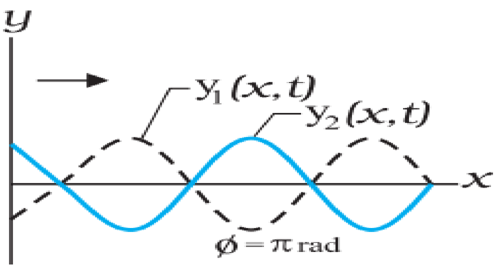


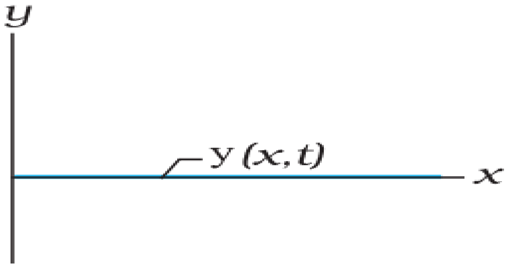
Resultant displacement, $y(x,t) = 2a \sin(kx - \omega t)$

Amplitude , $A = 2a$

which is the largest possible value of Amplitude A.

If $\phi = \pi$, i.e. the two waves are in phase,



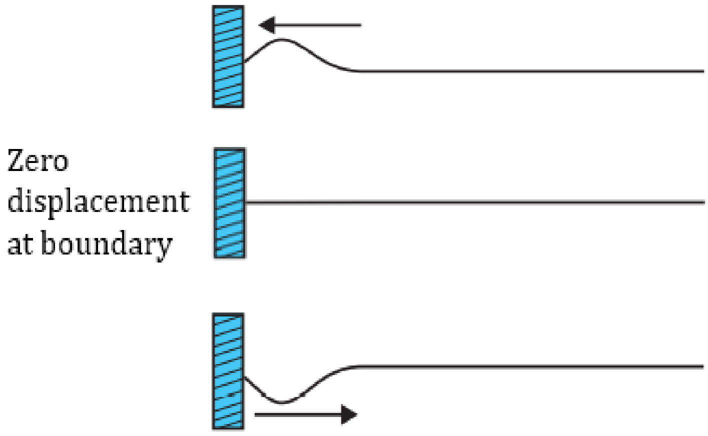


$$y(x, t) = 0$$

$$A = 0$$

Reflection of Wave

The reflection at a rigid boundary



- The reflected wave will have a phase reversal i.e, a phase difference of π radian or 180^0.
- There will be no displacement at the boundary as the string is fixed there.

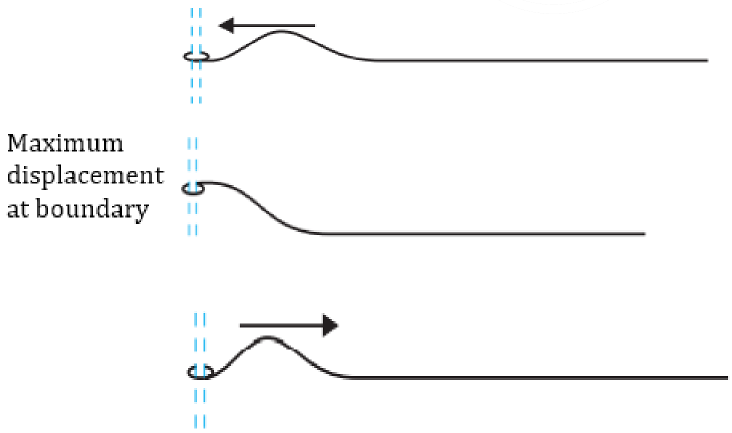
Incident wave, $\boldsymbol{y_i(x, t) = a \sin (kx - \omega t)}$

Reflected wave, $y_r(x, t) = a \sin (kx + \omega t + \pi)$

sin(180+θ)=-sin θ

$$\boldsymbol{y_r(x, t) = - a \sin (kx + \omega t)}$$

The Reflection at an Open Boundary



- The reflected wave will have same sign (no phase reversal) and amplitude as the incident wave.
- There will be maximum displacement at the boundary(twice the amplitude of either of the pulses)

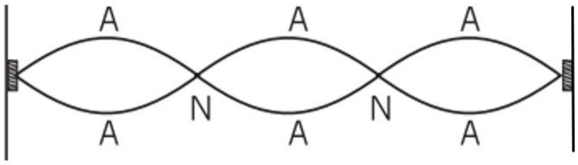
Incident wave, $\boldsymbol{y_i(x, t) = a \sin (kx - \omega t)}$

Reflected wave, $\boldsymbol{y_r(x, t) = a \sin (kx + \omega t)}$.

Standing Waves and Normal Modes

Standing Waves

The interference of two identical waves moving in opposite directions produces standing waves.



Wave travelling in the positive direction of x-axis

$$y_1(x, t) = a \sin (kx - \omega t)$$

Wave travelling in the negative direction of x-axis

$$y_2(x, t) = a \sin (kx + \omega t)$$

By the principle of superposition

$$y(x, t) = y_1(x, t) + y_2(x, t) = a \sin (kx - \omega t) + a \sin (kx + \omega t)$$

$$y(x, t) = (2a \sin kx) \cos \omega t$$

This equation represents a standing wave, a wave in which the waveform does not move.

Amplitude of wave , $A = 2a \sin kx$.

Nodes and Antinodes

The positions of zero amplitude in a staning wave are called nodes and the positions of maximum amplitude are called antinodes.

Condition for Nodes

At nodes,the amplitude of standing wave is zero

$$2a \sin kx = 0$$

$$\sin kx = 0$$

$$kx = n\pi, \text{ for } n = 0, 1, 2, 3, ..$$

$$\text{But } k = \frac{2\pi}{\lambda}$$

$$\frac{2\pi}{\lambda} x = n\pi$$

$$x = n\frac{\lambda}{2}, \quad \text{for } n = 0, 1, 2, 3, ..$$

i.e., nodes are formed at locations x=0, $\frac{1\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}$,... ......

The nodes are separated by $\lambda/2$ and are located half way between pairs of antinodes.

Condition for Antinodes

At antinodes, the amplitude of standing wave is maximum.

$$2a \sin kx = \text{maximum}$$

$$\sin kx = \pm 1$$

$$kx = \left(n + \frac{1}{2}\right)\pi, \text{ for } n = 0, 1, 2, 3, ..$$

$$\text{but}, k = \frac{2\pi}{\lambda}$$

$$\frac{2\pi}{\lambda} x = \left(n + \frac{1}{2}\right)\pi$$

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \quad , \quad \text{for } n = 0, 1, 2, 3, ..$$

i.e., antinodes are formed at locations x= $\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ **,.........**

The antinodes are separated by $\lambda/2$ and are located half way between pairs of nodes.

(1)Standing waves in a Stretched String fixed at both the ends

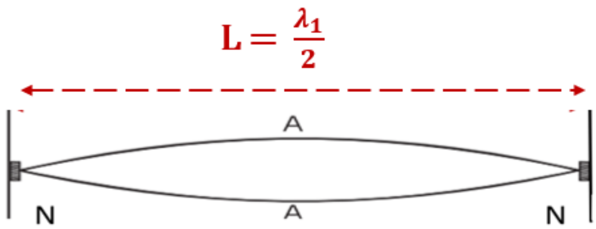
For a stretched string of length L, fixed at both ends, the two ends x= 0 and x= L of the string have to be nodes.

The condition for node at L

$$L = n\frac{\lambda}{2}, \quad \text{for } n = 1, 2, 3, ...$$

Fundamental mode or the first harmonic

The oscillation mode with n=1 , the lowest frequency is called the fundamental mode or the first harmonic.



$$L = \frac{\lambda_1}{2}$$

$$\lambda_1 = 2L$$

$$\text{But } v = \nu\lambda,$$

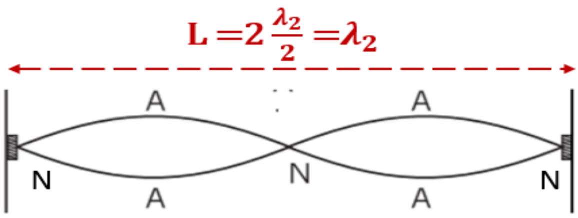
$$\nu = \frac{v}{\lambda}$$

$$\text{Frequency, } \nu_1 = \frac{v}{\lambda_1}$$

$$\nu_1 = \frac{v}{2L} \quad \text{----------(1)}$$

The second harmonic

The second harmonic is the oscillation mode with n = 2.



$$L = 2\frac{\lambda_2}{2} = \lambda_2$$

$$\lambda_2 = L$$

$$\text{Frequency, } \nu_2 = \frac{v}{\lambda_2}$$

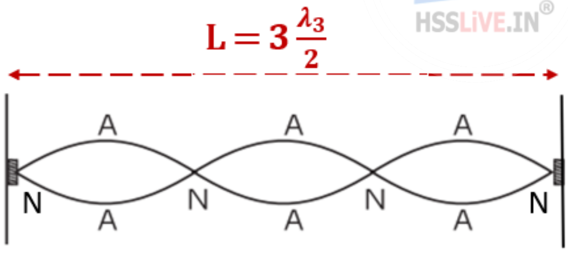
$$\nu_2 = \frac{v}{L}$$

$$\nu_2 = 2\frac{v}{2L} \text{ -------------(2)}$$

$$\nu_2 = 2\nu_1$$

The Third Harmonic

The third harmonic is the oscillation mode with n = 3.



$$L = 3\frac{\lambda_3}{2}$$

$$\lambda_3 = \frac{2L}{3}$$

$$\text{Frequency, } \nu_3 = \frac{v}{\lambda_3}$$

$$\nu_3 = \frac{v}{\frac{2L}{3}}$$

$$\nu_3 = 3\frac{v}{2L} \text{ --------------(3)}$$

$$\nu_3 = 3\nu_1$$

and so on.

$$\nu_1 : \nu_2 : \nu_3 = 1 : 2 : 3$$

Thus all harmonics are possible in a stretched string fixed at both the ends.

(2) The modes of vibration in a closed pipe (system closed at one end and the other end open).

Eg: Resonance Column(Air columns such as glass tubes partially filled with water) .

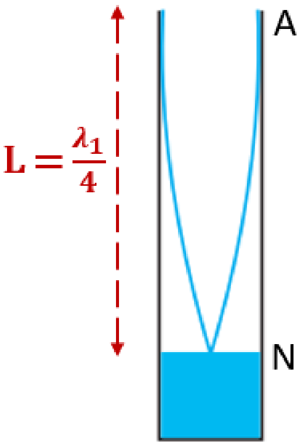
If the length of the air column is L, then closed end x=0 is a node and the open end, x = L, is an antinode.

The condition for antinode at L

$$L=\left(n+\frac{1}{2}\right)\frac{\lambda}{2} \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Fundamental mode or the first harmonic

The oscillation mode with n=0 , fundamental mode or the first harmonic.



$$L = \frac{\lambda_1}{4}$$

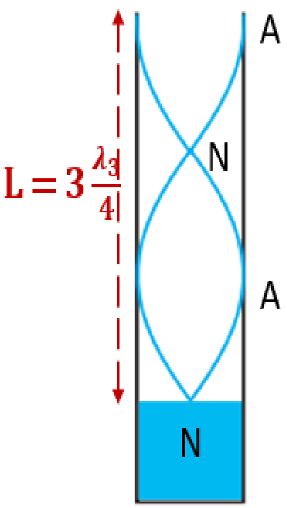
$$\lambda_1 = 4L$$

$$\text{Frequency, } \nu_1 = \frac{v}{\lambda_1}$$

$$\nu_1 = \frac{v}{4L} \text{ ------------(1)}$$

The Third Harmonic

The Third harmonic is the oscillation mode with n = 1.



$$L = 3\frac{\lambda_3}{4}$$

$$\lambda_3 = \frac{4L}{3}$$

$$\text{Frequency, } \nu_3 = \frac{v}{\lambda_3}$$

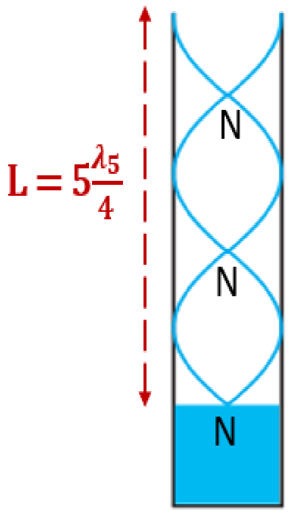
$$\nu_3 = \frac{v}{\frac{4L}{3}}$$

$$\nu_3 = 3\frac{v}{4L} \text{ --------------(2)}$$

$$\nu_3 = 3\nu_1$$

The Fifth Harmonic

The Fifth harmonic is the oscillation mode with n = 2.



$$L = 5\frac{\lambda_4}{4}$$

$$\lambda_4 = \frac{4L}{5}$$

$$\text{Frequency, } \nu_5 = \frac{v}{\lambda_5}$$

$$\nu_5 = \frac{v}{\frac{4L}{5}}$$

$$\nu_5 = 5\frac{v}{4L}$$

$$\nu_5 = 5\nu_1$$

And so on.

$$\nu_1 : \nu_3 : \nu_5 = 1 : 3 : 5$$

Thus only odd harmonics are possible in a closed pipe.

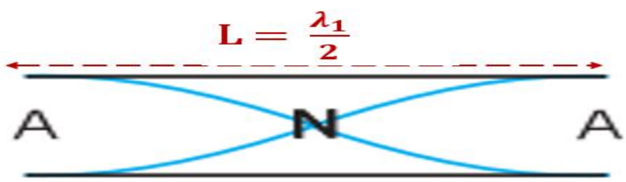
(3) The modes of vibration in a an open pipe (system open at both ends). Eg: Flute

For an open pipe of length L, antinodes are formed at both ends.

$$L = n\frac{\lambda}{2}, \quad \text{for n} = 1, 2, 3, \ldots$$

Fundamental Mode or The First Harmonic

The oscillation mode with n=1 , the lowest frequency is called the fundamental mode or the first harmonic.



$$L = \frac{\lambda_1}{2}$$

$$\lambda_1 = 2L$$

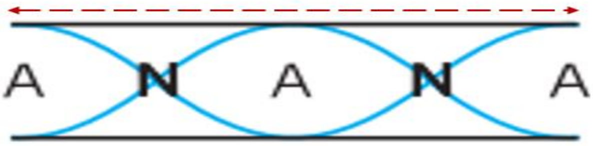
$$\text{Frequency, } \nu_1 = \frac{v}{\lambda_1}$$

$$\nu_1 = \frac{v}{2L} \text{------------}(1)$$

The Second Harmonic

The second harmonic is the oscillation mode with n = 2.

$$L = 2\frac{\lambda_2}{2} = \lambda_2$$



$$L = 2\frac{\lambda_2}{2}$$

$$\lambda_2 = L$$

$$\text{Frequency, } \nu_2 = \frac{v}{\lambda_2}$$

$$\nu_2 = \frac{v}{L}$$

$$\nu_2 = 2\frac{v}{2L} \text{------------(2)}$$

$$\nu_2 = 2\nu_1$$

The Third Harmonic

The third harmonic is the oscillation mode with n = 3.

$$L = 3\frac{\lambda_3}{2}$$

$$L = 3\frac{\lambda_3}{2}$$

$$\lambda_3 = \frac{2L}{3}$$

$$\text{Frequency, } \nu_3 = \frac{v}{\lambda_3}$$

$$\nu_3 = \frac{v}{\frac{2L}{3}}$$

$$\nu_3 = 3\frac{v}{2L} \text{------------(3)}$$

$$\nu_3 = 3\nu_1$$

and so on.

$$\nu_1 : \nu_2 : \nu_3 = 1 : 2 : 3$$

Thus all harmonics are possible in an open pipe.

So open pipes are preferred over closed pipes in musical instruments.

Beats

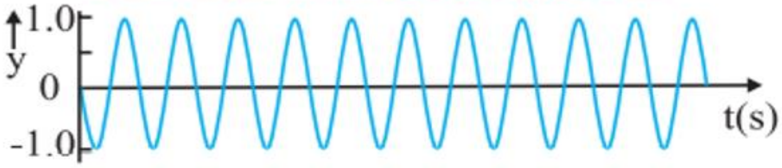
The periodic variations(wavering) of sound intensity when two waves of nearly same frequencies and amplitudes travelling in the same direction, are superimposed on each other is called beats.

These wavering of sound is also called waxing and waning.

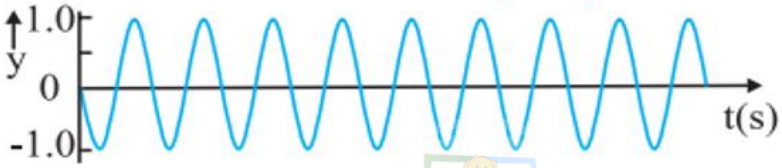
If ν_1 *and* ν_2 are the frequencies of superposing waves, the beat frequency

$$\boldsymbol{\nu_{beat} = \nu_1 - \nu_2}$$

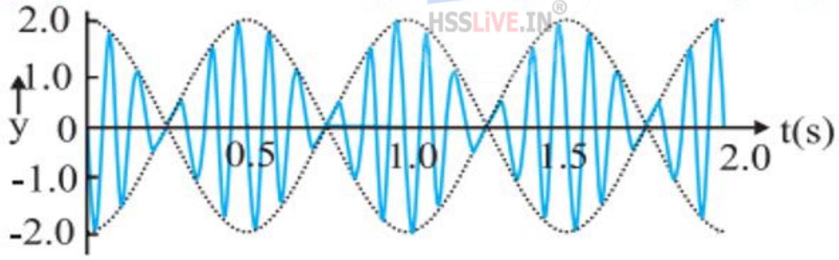
A harmonic wave of frequency 11 Hz.



A harmonic wave of frequency 9 Hz



Superposition of two wave, producing beats of frequency 2Hz



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